7.7.1. Aspects of adequacy

What we have been asking of our system of derivations is that it always give us the right answer concerning the validity of a conclusion. But it was noted already in 2.3.3 that we would eventually have to retrench and ask only that the system be ***complete*** (in the sense of giving all correct affirmative answers) and ***sound*** (in the sense of never giving incorrect affirmative answers). A system that is complete and sound thus tells us that an argument is valid when and only when it is valid. Since completeness implies that we never get an incorrect negative answer, the two properties together also imply that, while we may not get all the right answers, all the answers we get will be right.

They also imply that we can fail to get an answer only when the answer is negative. That sort of asymmetry is possible only if we can be in the position of not knowing whether we will ever receive an answer, for otherwise we could interpret silence as dissent. But that's just the position we are in if the process of developing a gap never ends. If a system of derivations is not decisive, we may not know in advance whether we will eventually get an answer. And, if not, we are in the position of someone waiting for a door or a phone to be answered: one more knock or ring may be enough but, then again, we might not receive an answer no matter how long we wait. Similarly, if we are working on a derivation that gives no answer, all we may ever know is that we have not received an answer yet.

Later, we will look more closely at why this can happen with derivations. But, first, we will see what can be salvaged from the sort of argument we have for the adequacy of the systems of previous chapters in order to show that our current system is at least sound and complete. In the approach taken in 2.3.2 and extended to the systems of later chapters, we argue for the soundness of a system solely on the basis of the (minimal) soundness of its rules. If any interpretation that divides the initial gap of a derivation continues to divide some gap at each stage in its development, a derivation whose initial gap is divided by some interpretation can never have all its gaps close. It follows, then, that if all gaps close, the initial entailment holds. This argument can be carried over to the system of derivations for generalizations if we can show that the rules for universal quantifiers are minimally sound. That is not hard to do, but we will need to refine our definition of soundness in order to accommodate rules that introduce new vocabulary into a derivation.

We saw in 2.3.3 how an argument for the completeness of the systems of chapters 2-6 can be based on the properties of decisiveness, conservativeness, and sufficiency. It was noted there that, to show completeness, we do not need the full property of decisiveness: we need to know only that we receive an answer about validity whenever the argument is valid. For conservativeness and sufficiency imply that any answer we receive about validity is correct, so, if we always receive some answer when an argument is valid, we can be sure that our system will recognize the validity of any valid argument. In order to establish this sort of semi-decisiveness, we need to show that, whenever a derivation develops infinitely without producing any dead-end gaps, a negative answer to the question of validity is the correct one. This requires a different sort of argument from that used to show that any dead-end open gap establishes the existence of a counterexample, but the difference is not great, and the new argument will apply also to dead-end gaps.

It will be easier to state this new argument if we extend the genealogical metaphor we have used to describe the development of derivations. Let us speak of a line of descent from parents to children to grandchildren, etc., as a ***path***. A path always begins at the initial stage of the derivation and ends only when the last gap of the path has no children. A path at a given stage may be developed at the next stage by adding a child of its last gap; if there is more than one child it will divide into two or more paths as it develops. We will say that an interpretation ***divides*** a path when it divides each gap in the path.

If we think of a path as it develops through time, we can imagine a path within which any applicable rule is eventually applied even though that path never ends. Each way of developing such a path will be employed at some point, but there will be no point at which there is nothing more to be done. So let us say that a path ***develops fully*** if the path never closes but all that could be done to develop it is done at some point in its development. Such a path may end with a dead-end gap, but it need not. We can use the safety of rules and ideas from arguments for sufficiency to show that any gap that develops fully is divided by an interpretation. We will say that a system with this property is ***effectual***.

Since every path stems from the initial gap of the derivation, if we are able to divide a fully developing path, we will know that we can divide the initial gap and that a negative answer to the question of validity is the correct one. This means that we will be able to establish completeness for an effectual system if we can show also that any derivation that does not close will have some path that develops fully. Let us say that a system for which this is true is ***thorough***.

To recap, we may show that our system is sound by showing that its rules are (minimally) sound. And we may show that it is complete by showing that it is thorough and effectual. This is summarized in Table 7.7.1-1.

rules are minimally *sound:* they never drop interpretations that divide the initial gap	⇒	system is *sound:* if all gaps close, entailment holds
system is *effectual:* any fully developing path is divided by an interpretation	} ⇒	system is *complete:* if entailment holds, all gaps close
system is *thorough:* development is organized so that either all gaps close or at least one path develops fully		

Table 7.7.1-1. Some logical relations among properties of a system of derivations. (The brace indicates that the second entailment has two premises.)

Although our system of derivations for universals is not decisive, it is sound and complete. And that makes it pretty good, especially since the positive use of derivations is the more important one. But why should a pretty good system be good enough? The answer is that we cannot do any better. There can be systems that answer questions in cases where ours is silent, but there is none that will answer in all cases and never answer incorrectly. This was shown in the mid-1930s by Alonzo Church (the originator of the lambda notation) based on work a few years earlier by Kurt Gödel (who, a little earlier still, was the first to establish the completeness of an account of validity for arguments involving generalizations).

There cannot even be a system that picks up where ours leaves off by giving all correct negative answers and never giving incorrect ones. The argument here is easy once it is shown that no system can be found that is both decisive and accurate: if there were a system that complemented ours, we could make a system that was decisive by using ours and its complement in tandem since, no matter what question we asked, one or the other system would eventually give us an answer.

We will go on to look in more detail at the virtues our system does have. First we will re-define soundness and consider the soundness of rules for universals. Next we will see what is takes to insure thoroughness. After that we will look at the argument for effectuality. We will return to the negative side of things in section 7.8, where we will look more closely at the reasons why decisiveness fails.