7.6.1. Principles for restricted universals

The laws for unrestricted universals as premises and as conclusions were based on the relation between a generalization and its instances. In the case of restricted universals, we will also be interested in instantiation, but an unrestricted universal $(\forall x\colon \rho x)\ \theta x$ does not in general imply an instance $\theta\tau$. Still, since $(\forall x\colon \rho x)\ \theta x \Leftrightarrow \forall x\ (\rho x \to \theta x)$, the restricted universal will imply the sentences $\rho\tau \to \theta\tau$ that are the instances of its restatement in unrestricted form; we will call these its ***conditioned instances***. If we expand the language by the range **R** of a structure, a restricted universal $(\forall x\colon \rho x)\ \theta x$ will be true in that structure if and only if all its conditioned instances are true. That means that a restricted universal behaves like a conjunction of its conditioned instances, and its role in deductive reasoning is analogous in some respects to the role of an unrestricted universal and in other respects to the role of a conditional.

We can get laws for the restricted universal by restating such a universal in unrestricted form and applying laws for both the unrestricted universal and the conditional. To get our ***law for the restricted universal as a conclusion***, we can reason as follows for any term a that does not appear in Γ or $(\forall x\colon \rho x)\ \theta x$:

$\Gamma \Rightarrow (\forall x\colon \rho x)\ \theta x$	
if and only if	
$\Gamma \Rightarrow \forall x\ (\rho x \to \theta x)$	*restatement with an unrestricted quantifier*
if and only if	
$\Gamma \Rightarrow \rho a \to \theta a$	*by the law for a universal as a conclusion*
if and only if	
$\Gamma, \rho a \Rightarrow \theta a$	*by the law for a conditional as a conclusion*

That is, we can conclude a restricted universal if and only if we can conclude its instance θa for a parametric term a, allowing ourselves to make the assumption that ρa—i.e., that the value of the term a is in the domain of the universal. The assumption here that ρa is comparable to the assumption that *ABC* is a triangle which is made when we wish to offer an argument about *ABC* as a basis for a generalization about all triangles.

We can approach the role of $(\forall x\colon \rho x)\ \theta x$ as a premise also by way of a restatement in unrestricted form. If we apply the law for an unrestricted universal premise and restate the result again using a restricted quantifier, we get, for any term τ,

$$\Gamma, (\forall x\colon \rho x)\ \theta x \Rightarrow \phi \text{ if and only if } \Gamma, (\forall x\colon \rho x)\ \theta x, \rho\tau \to \theta\tau \Rightarrow \phi$$

To go further, we need to take account of the conditional premise we have introduced. We have three ways of doing this, two detachment principles, each of which requires still another premise, and a principle for *reductio* arguments. Accordingly, we get three principles for a restricted universal premise each applying to a different sort of case:

Γ, $(\forall x\colon \rho x)\ \theta x$, $\rho\tau \Rightarrow \phi$	*if and only if*	Γ, $(\forall x\colon \rho x)\ \theta x$, $\rho\tau$, $\theta\tau \Rightarrow \phi$
Γ, $(\forall x\colon \rho x)\ \theta x$, $\overline{\theta\tau} \Rightarrow \phi$	*if and only if*	Γ, $(\forall x\colon \rho x)\ \theta x$, $\overline{\theta\tau}$, $\overline{\rho\tau} \Rightarrow \phi$
Γ, $(\forall x\colon \rho x)\ \theta x \Rightarrow \bot$	*if and only if*	both Γ, $(\forall x\colon \rho x)\ \theta x \Rightarrow \rho\tau$ and Γ, $(\forall x\colon \rho x)\ \theta x$, $\theta\tau \Rightarrow \bot$

Each holds for any term τ.

The first two are related to the following valid forms of argument in the way that the detachment rules for conditionals are related to the argument patterns *modus ponens* and *modus tollens*.

Singular Barbara

$$\frac{(\forall x\colon \rho x)\ \theta x \qquad \rho\tau}{\theta\tau}$$

Singular Camestres

$$\frac{(\forall x\colon \rho x)\ \theta x \qquad \overline{\theta\tau}}{\overline{\rho\tau}}$$

The first argument provides a sort of ***restricted universal instantiation***. It is perhaps the most widely recognized pattern of argument; an instance of it was our first example of a valid argument in 1.1.2. In the logical tradition, it and the second pattern (which stands to it as *modus tollens* stands to *modus ponens*) were often not distinguished from certain patterns of argument whose second premises and conclusions contain quantifier phrases rather than individual terms, and the names used here are adapted from names for those arguments in a medieval system of nomenclature for syllogisms. (Notice that vowels in the names are the first vowels appearing in the English verbs *affirm* and *negate*, which happen to be cognate to Latin terms, and that these vowels mark the affirmative or negative character of the premises and conclusion taken in order. Many of the consonants in these names are also significant, pointing to connections among various patterns of argument in the theory of syllogisms proper.)

The third principle for restricted universal conclusions says that we can reduce a generalization $(\forall x\colon \rho x)\ \theta x$ to absurdity given Γ if and only if we can use a term τ to produce a counterexample to the generalization. We try to establish this counterexample by doing two things—(i) showing that the value of τ is in the domain of the generalization and (ii) reducing to absurdity the claim that the attribute of the generalization is true of the value of τ. The *if* part of this principle should be unsurprising since a counterexample to a generalization is inconsistent with it. But the full *if-and-only-if* claim may seem odd, since it implies that if one term will serve as a counterexample, so will any other. It is true that given premises might entail one counterexample to a generalization without entailing others, but if premises entail *both* a counterexample *and* and the generalization itself (as they certainly do if the generalization is among them), they form an inconsistent set and entail every sentence. Of course, we may be able to show that one term provides a counterexample only by first showing that another one does. Since, unlike the principles of Singular Barbara and Singular Camestres, this principle applies to any *reductio* argument that has a generalization as a premise, we will count it as our general ***law for the restricted universal as a premise***.

To summarize, we have the following basic principles for the restricted universal:

Law for the restricted universal as a premise. For any term τ, we have:

$$\Gamma, (\forall x\text{: } \rho x)\ \theta x \Rightarrow \bot \text{ if and only if both } \Gamma, (\forall x\text{: } \rho x)\ \theta x \Rightarrow \rho\tau \text{ and } \Gamma, (\forall x\text{: } \rho x)\ \theta x, \theta\tau \Rightarrow \bot.$$

Law for the unrestricted universal as a conclusion. For any unanalyzed term a appearing in neither Γ nor (∀x: ρx) θx, we have:

$$\Gamma \Rightarrow (\forall x\text{: } \rho x)\ \theta x \text{ if and only if } \Gamma, \rho a \Rightarrow \theta a.$$

Although these two principles suffice to capture the logical properties of restricted universals, rules implementing singular Barbara and singular Camestres will play an important role in our system of derivations.