7.5. General arguments

7.5.0. Overview

We have answered questions about entailment concerning truth-functional compounds by turning them into questions about their immediate components (or sentences contradictory to them). The largest component formulas of sentences formed by quantifiers usually contain free variables, so we will look at the sentences that are the result of putting closed terms in place of these variables.

7.5.1. Conjunction and universal quantification
An unrestricted universal sentence behaves like a conjunction of sentences saying of each particular thing what the universal says of everything.

7.5.2. The role of generalizations in entailment
The laws of entailment for unrestricted universals treat them as conjunctions of their instances; but they differ from the laws for conjunction itself in ways that reflect the fact that a universal has indefinitely many instances and that they are all predications of the same abstract.

7.5.3. Derivations for universals
Because a universal has indefinitely many instances, we cannot consider each in a derivation. Instead, we establish a universal by arguing about a single "typical" instance, and we exploit a generalization only partially to extract those instances that are relevant to the argument we are considering.

Glen Helman 06 Nov 2004

7.5.1. Conjunction and universal quantification

The truth conditions of generalizations are analogous to those of conjunctions. So, before looking at laws and rules for the universal quantifiers, we will spend some time comparing these operations to conjunction.

Consider the pair of sentences analyzed below.

Every permanent member of the Security Council supported the resolution

(∀x: Mxs) Sxl

Britain, China, France, Russia, and the U. S. supported the resolution

Sbl ∧ Scl ∧ Sfl ∧ Srl ∧ Sul

[M: λxy (x *is a member of* y); S: λxy (x *supported* y); b: *Britain*; c: *China*; f: *France*; l: *the resolution*; r: *Russia*; s: *the Security Council*; u: *the U. S.*]

These two sentences have the same truth value, but they are not equivalent because in a different possible world the membership of the Security Council could be different. However, consider the sentence

Each of Britain, China, France, Russia, and the U. S. supported the resolution

This could be analyzed in the same way as the second sentence above, but it could be analyzed also as a restricted universal whose restricting predicate is λx (x *is Britain, China, France, Russia, or the U. S.*)—switching to *or* here for the same reasons that lead to us switch in handling *all boys and girls* (see 7.3.2). A full analysis would give us the following:

(∀x: x=b ∨ x=c ∨ x=f ∨ x=r ∨ x=u) Sxl

And this universal *is* equivalent to the conjunction because either way we say that the predicate λx (x *supported* l) is true of the reference values of b, c, f, r, and u.

Each of the universals (∀x: ρx) θx and ∀x θx says that the predicate θ is true of each value in the domain over which it generalizes. Only in special cases (like the example just above) will either be equivalent to a conjunction

$$\theta\tau_1 \wedge \theta\tau_2 \wedge \ldots \wedge \theta\tau_n$$

that predicates θ of each of a series of terms. But it can still be enlightening to compare universals to such conjunctions, so we will develop some vocabulary for doing so. In this section, we will do this only for unrestricted universals, turning to the case of restricted universals in 7.6. Let us say that an ***instance for*** a term τ of a universal ∀x θx is a sentence θτ that applies the quantified predicate θ to τ—that is, an instance of a universal ∀x ... x ... has the form ... τ ..., the result of putting τ in place of the occurrences of that variable x that are bound to the quantifier ∀x. An instance asserts of a single reference value what the universal asserts of everything in its domain.

If every reference value is the extension of some term, an unrestricted universal ∀x θx will be true if and only if each of its instances θτ is true. This means that it will behave like a conjunction of these instances. But this is not we could work with such a conjunction in place of the universal because, given just one unanalyzed term and one functor, there will be infinitely many compound terms and infinitely many instances of any universal whose quantifier actually binds a variable. For example, given an unanalyzed term a and functor f, the language will contain the terms

a, fa, f(fa), f(f(fa)), ...

and a universal ∀x Px will have the instances

Pa, P(fa), P(f(fa)), P(f(f(fa))), ...

Although it is possible to make sense of infinite conjunctions as part of a mathematical structure if there is no expectation that it be possible to write them down, our references to conjunctions of all instances will be only a figure of speech used to motivate and guide our treatment.

For an unrestricted universal to behave like a conjunction of its instances, every reference value must be the value of some term. So let us develop the figure of speech further by imagining that the ID of each reference value in a range **R** is added as a further term of our language. We will speak of this operation as ***expansion by* R**. If we expand the language by the range **R** of a structure, an unrestricted universal ∀x θx will be true in that structure if and only if all its instances are true.

7.5.2. The role of generalizations in entailment

The special features of the laws we will state for the universals can be traced to two sources. One is the analogy with conjunction we have just explored. The other is a pair of differences between what we have said about universals and what we may say about ordinary conjunctions. The first difference lies in the fact that the principles of entailment for universals must hold for all structures, so they cannot depend on special assumptions about the range **R** of reference values. This means, in particular, that the set of "components" of a universal (i.e., its instances in an expansion by **R**) must be left indefinite while an ordinary conjunction has a definite and, indeed, finite set of components. This would make universals difficult to deal with were it not for their second difference from conjunctions. The components of an ordinary conjunction can be any pair of sentences so they need have nothing in common, and we must consider them individually; but the instances of a universal all follow the same pattern, differing only in occurrences of a single term, so we can speak of them all together by speaking of this pattern.

We will develop laws for universals by taking certain laws for conjunctions as our model and modifying them to take account of the differences just outlined. The laws for conjunction we will work from are the following:

$$\phi \wedge \psi \Rightarrow \phi$$
$$\phi \wedge \psi \Rightarrow \psi$$
$$\Gamma \Rightarrow \phi \wedge \psi \text{ if and only if } \Gamma \Rightarrow \phi \text{ and } \Gamma \Rightarrow \psi.$$

The last should be no surprise since it is our basic law for conjunction as a conclusion but, although the first pair of principles are clearly associated with the rule of Extraction, they are less far reaching than our law for conjunction as a premise. Our use of these less general principles is itself due to the differences between universals and conjunction: the law for conjunction as a premise says we can replace a conjunction by its components, but there is no hope of doing anything like this for a universal since it has no one definite set of "components."

What laws can we state for unrestricted universals that are analogous to these laws for conjunction? The first pair of laws for conjunction noted above together say that a conjunction implies each of its components. The analogous claim about an unrestricted universal is that it implies each of its instances. This is a principle known as ***universal***

instantiation:

$$\forall x\ \theta x \Rightarrow \theta\tau \text{ for each term } \tau. \text{ (That is, } \forall x \ldots x \ldots \Rightarrow \ldots \tau \ldots\text{.)}$$

For example, the sentence *Everything is fine and dandy* implies the claim *The number 2 is fine and dandy* as well as other sentences of the form τ *is fine and dandy*.

The principle of universal instantiation is not quite what we will take as our account of the unrestricted universal as a premise. Universal instantiation can be used along with the law of lemmas to develop a derivation by adding any instance of a universal premise as a further resource. And it is that use of universal instantiation that we call our ***law for the unrestricted universal as a premise***: for any term τ, we have

$$\Gamma, \forall x\ \theta x \Rightarrow \Sigma \text{ if and only if } \Gamma, \forall x\ \theta x, \theta\tau \Rightarrow \Sigma.$$

That is, given a universal premise, we may add any instance as a further premise. Note that the instance is added as a *further* premise. We cannot drop the universal because we cannot expect its content to be exhausted by a single instance; *Everything is fine and dandy*, for example, has implications for things other than the number 2. As you might expect, our inability to drop the universal from the premises will some cause complications when we try to implement this law in derivations.

Now let us look at the role of an unrestricted universal as a conclusion. Here we have the law for conjunction as a conclusion to use as a model. We have to expect changes, though, because that law gives separate consideration to each of the two components of the conjunction and we cannot expect to do this for the instances of a universal. Still, the law for conjunction points us in the right direction: we should look for some way of connecting the validity of a universal conclusion with the validity of arguments having its instances as conclusions. A connection like this is used in geometric proofs when we begin by saying, for example, "Let *ABC* be a triangle," and then go on to use our conclusions concerning *ABC* to justify general conclusions about all triangles. That is, we sometimes establish universal claims by ***generalizing*** from particular instances of them.

Clearly not every generalization from a particular instance will be legitimate. Certain premises may entail *The Empire State Building is tall* without entailing *Every building is tall*. In a geometric argument concerning a triangle *ABC*, we limit the information that we may use about the instance that we are considering to what we may establish concerning any triangle. For example, we ignore the fact we are using a diagram that shows *ABC* as acute or obtuse, and we probably avoid drawing it as a right triangle or an isosceles triangle to begin with. These restrictions are sometimes expressed by saying that we are arguing about an *arbitrary* or an *arbitrarily chosen* triangle. The idea is that what you say about the triangle *ABC* should hold for a triangle chosen at random or even one chosen by your worst enemy. Let us call an argument like this a ***general argument*** since it argues for an instance in a way that will hold generally for values in the domain of a universal. The law we are looking for should say that an unrestricted universal is a valid conclusion from given premises if we can establish an instance of it by a general argument. But we need to make this more precise.

In particular, we need to say how we can recognize a general argument just by looking at the logical forms of the sentences it involves. If we were to give instructions for making a general argument about a triangle *ABC*, one thing we might say is that we should not use any special assumptions about *ABC*. If we are going to generalize about triangles, we may assume that *ABC* is a triangle but we should not assume that it is acute or obtuse. This is just another way of saying that we should not use special information about this triangle, but it suggests an idea we can apply to arguments when we know only their logical forms.

Since we are considering arguments for unrestricted universals, we must be able to generalize not just about triangles, or some other limited class, but about everything; and that means we should use no assumptions at all about the term from which we wish to generalize. So we can say this: if we wish to generalize from an instance θτ to a universal ∀x θx, the term τ should not appear in our assumptions. You may have noticed a couple of jumps here. Saying we have an assumption containing τ is different from saying we have used that assumption, and saying that τ appears in an assumption is different from saying that the assumption provides special information about τ. For example, *The number 2 is fine and dandy and so is everything else* mentions the number 2 without constituting a special assumption about it. Still, the requirement that the term from which we generalize not appear in the assumptions is easy to check and using it will not limit the entailments we can establish, only the terms we can use to establish them.

This requirement is enough to rule out many unwarranted generalizations but it does not exclude them all. To see why, suppose we are arguing from the assumption *Everything is like itself*. One conclusion we can draw is *Wabash is like Wabash* and, in doing so, we have certainly used no special assumptions about Wabash. But this conclusion says that Wabash has the property of being like Wabash and that makes it an instance of the generalization *Everything is like Wabash*. But generalizing to that conclusion is surely unwarranted. The problem with this argument is that even though the term *Wabash* stands in no special relation to the assumptions, it does stand in a special relation to the universal conclusion *Everything is like Wabash*. In particular, it plays a special role in the predicate that the conclusion claims to be universal. These considerations suggest a second requirement for a general argument: if we wish to generalize from an instance θτ to a universal ∀x θx, the term τ should not appear in our conclusion; that is, it should not appear in θ.

There remains only one sort of problem to consider. Suppose our assumption is *Everything has its bad side*. We can conclude *Wabash has its bad side*. But we cannot go on to conclude *Wabash has everything*. Now the instance from which this conclusion would generalize is an instance for the term *Wabash's bad side* and this term does not appear in either the assumption or the conclusion, so it satisfies both of the requirements we have imposed so far. We could handle cases like this by requiring that terms on which we generalize share no vocabulary with either the assumptions or the conclusion. That would take care of this case (since *Wabash's bad side* shares vocabulary with both) and it would be more than enough to insure that an argument is general. Indeed, it would be enough to require, of a compound term, that its main functor not appear in the assumptions or conclusion (so, in the example above, the real problem is the appearance of the functor λx (x*'s bad side*) in the premise and not the appearance of the term *Wabash* in the conclusion). However, it is easier simply to prohibit generalization on compound terms. Unanalyzed terms that satisfy the first two requirements clearly share no vocabulary with the assumptions or conclusion so, for those terms, the first two requirements are enough.

We are now ready to state our ***law for the unrestricted universal as a conclusion***: for any unanalyzed term a appearing in neither Γ nor ∀x θx, we have

$$\Gamma \Rightarrow \forall x\ \theta x \text{ if and only if } \Gamma \Rightarrow \theta a.$$

Let us say that an unanalyzed term appearing in neither the premises or conclusion of an argument is *parametric*, or a ***parameter***, for that argument. In this vocabulary, the law says that an argument with an unrestricted universal conclusion is valid if and only if the premises entail an instance of the universal for a parameter. When arguments are stated in English, phrases like *let* a *be arbitrary* or *let us choose* a *arbitrarily* function as commitments to use the term a as a parameter.

Can be we sure that there will always be a parametric term on hand when we need one? Clearly, we would be stymied if our premises contained every term in the language. That is not a practical concern, but it shows that if there is always to be an appropriate term available for even any finite set of premises, our language must contain infinitely many parametric terms. Can we be sure that it does? If we are working with an idealized model of English, we can just stipulate that is does. Parametric terms are needed only for the inner workings of a derivation and need never appear in the initial premises and conclusion, so they do not have to be already available in the language. But even in real English there seems to be no shortage. The letter X is certainly overworked, but mathematicians always seem able to find one more symbol, no matter how many they are already juggling. If our imaginations falter, we can always resort to X′, X″, X‴, and so on—or else, X_1, X_2, ...

There is a more crucial question about this law: is it really true? We have been adding restrictions to insure that generalization is warranted. Can we be sure that we have enough? To see that we do, note first that the *only-if* part of the law is no problem. It says that a universal cannot be a valid conclusion unless any instance for a parameter is also valid, and this must be so because the universal implies all its instances.

So, let us consider the *if* part. To establish it, it is easiest to show that failure of the entailment Γ ⇒ ∀x θx implies failure of the entailment Γ ⇒ θa when a is parametric. Now, for the first entailment to fail, it must be possible to find a reference value that serves as a counterexample to the universal ∀x θx in some case where each member of Γ is true. Since the parameter a is foreign to both the premises and the universal conclusion, it can be made to refer to such a counterexample without interfering with the truth values of the premises. But this is to say we can make θa false while all members of Γ are true, and that means that the entailment Γ ⇒ θa fails. Turning this around, if Γ ⇒ θa holds when a is a parameter, then, when the premises Γ are all true, we know that there is no counterexample to the universal, which implies that the universal is true.

There is a more concrete way of showing that a parametric term is enough to insure that we have a general argument. Suppose we have established a conclusion θa for a parametric term a. Generalization to ∀x θx will be legitimate if we can argue from the same premises to each of its instances θτ. But we can find an argument for an instance θτ by following the pattern set by the argument for θa; we can simply replace the term a by τ everywhere in the argument for θa to get an argument for θτ. Because a is a parameter, it is not in the premises (and thus shares no vocabulary with them), so there is no connection between a and the premises that would make the argument go through for it but not for another term τ. And, because the parameter a is not in the conclusion, it is not in θ, so replacing it by τ everywhere will leave θ unchanged and change θa into $\theta\tau$.

This argument recalls the comparison of the universal with conjunction. Since a conjunction can have any components, we must argue for each component individually and, since a conjunction has only two components, there is nothing to keep us from doing this. On the other hand, there would be no hope of providing a separate argument for each instance of a universal since, in general, there is no way of setting a limit on the number of instances it has. However, there is no need to consider each of these instances individually since they all have the same form, so an argument for one parametric instance can set the pattern for all of the rest.

Let us collect the two laws for the unrestricted universal before going on to see how to implement them in derivations.

> ***Law for the unrestricted universal as a premise***. For any term τ, we have:
>
> $$\Gamma, \forall x\, \theta x \Rightarrow \Sigma \text{ if and only if } \Gamma, \forall x\, \theta x, \theta\tau \Rightarrow \Sigma.$$
>
> ***Law for the unrestricted universal as a conclusion***. For any unanalyzed term a appearing in neither Γ nor $\forall x\, \theta x$, we have:
>
> $$\Gamma \Rightarrow \forall x\, \theta x \text{ if and only if } \Gamma \Rightarrow \theta a$$

Again notice the difference between range of instances mentioned in the two laws. In the first, τ can be any term while, in the second, a must be a parameter relative to the premises and the universal.

7.5.3. Derivations for universals

The implementation of the laws for universal quantifiers is fairly straightforward if we use derivations only in a positive way—i.e., use them only to show that entailments hold. Their use to show that entailments fail will be postponed until 7.6. We do need one general elaboration of our system of derivations that we will use to manage general arguments. The portion of a derivation that constitutes a general argument will be marked by a scope line that is ***flagged*** by the parametric term on which we generalize (as shown in Figure 7.5.3-1). This flagging declares that the term is parametric. Indeed, we will require that a term flagging a scope line appear only to its right, so the scope line will mark the scope of the term's use. This is more than is necessary to stay in accord with the laws for universals as conclusions. They require only that the term not appear in either the goal or the active resources of the gap that the vertical line spans, but we will never run short of terms and the stronger requirement is far easier to check.

In either form, the requirement is designed to insure that the parameter maintains no ties to the outside of the general argument so that, within the argument, it might refer to anything at all. For this reason, we will speak of a scope line flagged by a term as a ***veil of ignorance***; the portion of the derivation marked off by the scope line proceeds without any information about the specific identity of the parameter a.

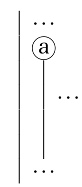


Fig. 7.5.3-1. A veil of ignorance flagged by the parameter a.

Now, let us look at the rules—first those for unrestricted quantifiers. Figure 7.5.3-2 shows the exploitation rule, which we will call ***Universal Instantiation*** (UI). It can be used to add any instance of the universal as a further resource, notating the universal to indicate the term for which an instance was added.

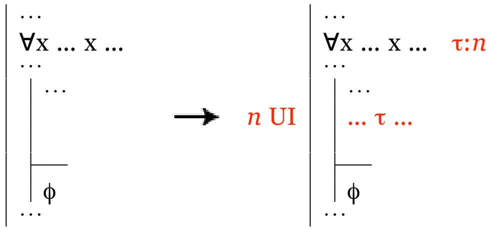


Fig. 7.5.3-2. Developing a derivation at stage *n* by exploiting an unrestricted universal for a term τ.

Although we record the use of this rule alongside the universal, the universal resource is not rendered completely inactive. The rule provides only a partial exploitation, extracting the content of the universal only for the single term τ. Since a universal does not bring with it any definite set of instances, it will never be rendered completely inactive, no matter how often this rule is used. Still, one use of the rule does exploit the universal for one term, and we record this by noting both the stage number and the term for which the universal has been exploited.

This information is used (much in the way we have used marking by stage numbers) to judge when a universal is ***active for*** a given term. To be active for a given term in a gap, a universal must be available in the gap and must not have been exploited for the term in the course of narrowing the gap. Specifically, an available universal is ***inactive for*** τ in a gap if it is marked by a pair τ:n and all scope lines to the left of some resource or goal entered at stage *n* continue unbroken to the left of the gap. Although an available universal is always active, it may not be active for all terms; and a term for which we apply the exploitation rule above should be one for which the universal is still active.

As we will see in 7.7.4, the use of this rule may be limited to terms appearing in the available resources and goals the gap. These are the same terms from which we form alias sets and it will be enough to exploit a universal for at least one term from each alias set. But, occasionally, no terms will appear in the initial premises and conclusion and none will be introduced by other rules. When this is so, the exploitation rule above may be used to introduce a new unanalyzed term into the derivation.

At the other extreme, use of this rule in case of generalizations containing functors may introduce new terms into the derivation, leading to new uses of the rule. For example, instantiating ∀x P(fx) for the term a will give us P(fa), which contains the term fa, and we may use this term also to instantiate the generalization, giving us P(f(fa)), which contains the term f(fa)—and so on. As we will see in 7.6, this is one aspect of a general feature of the deductive logic for generalizations that will sometimes keep a derivation from ever reaching an end. That is not our concern now, but the possibility of going on forever in the application of rules shows that we can no longer wait to apply rules fully before checking to see if a gap closes. And, because a large number of applications of instantiation may be possible, it is wise to select from among the terms with which we might instantiate a generalization those that seem most likely to help us close a gap.

The following two derivations illustrate these points.

	∀x ∀y ∀z Rxyz	a:1
1 UI	∀y ∀z Rayz	b:2, c:4
2 UI	∀z Rabz	c:3
3 UI	Rabc	(6)
4 UI	∀z Racz	c:5
5 UI	Racc	(6)
6 Adj	Rabc ∧ Racc	(7)
	•	
7 QED	Rabc ∧ Racc	

	∀x Fx	a:1
	∀x ¬ Fx	a:2
1 UI	Fa	(3)
2 UI	¬ Fa	(3)
	•	
3 Nc	⊥	

The first derivation keeps the use of UI to a minimum. Only the main quantifier is removed with each use, so three uses are required to reach the bare predication Rabc. Only two more are needed to reach Racc but three would have been required to reach a predication, such as Rccc, which did not have the term a in the first place after R. On the other hand, a full use of instantiation for the terms appearing in the conclusion would have lead to 3 + 3×3 + 3×3×3 = 39 uses of UI (i.e., three to exploit the premise for a, b, and c, three each to exploit the three resources that result, and three each to the nine resources added in that way). A derivation is not damaged by extra uses of UI any more than it is damaged by using Ext to add conjuncts that are not needed later. But, while adding all conjuncts as resources whenever a conjunction was exploited presented no practical problem, using UI in all ways possible would often lead to unmanageably large derivations.

The premises and conclusion of the second derivation above contain no terms at all, so there would be no way of beginning it if we did not instantiate one of them for a new term. This is the only sort of case in which instances need be added for terms new to the gap being developed. The fact that we do so at all reflects the assumption built into our system that there is at least one reference value. The derivation above shows one consequence of this assumption—namely, that *Everything is finished* and *Everything is unfinished* are inconsistent.

Clearly, if there is anything at all, then these two sentences cannot both be true. On the other hand, if we were to drop the assumption that there is something, both sentences could be true. For generalizations are false only when they have counterexamples; and, in a world in which there was nothing, there would be nothing to serve as a counterexample to either *Everything is finished* or *Everything is unfinished*.

Finally, let us look at the planning rule for universal goals. It is known as ***Universal Generalization*** (UG) and is shown in 7.5.3-3.

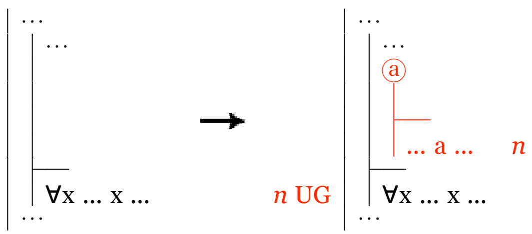
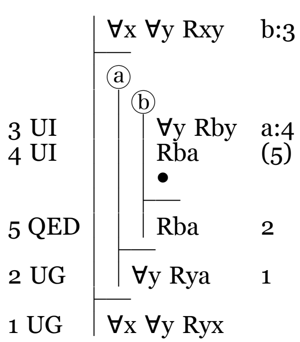


Fig. 7.5.3-3. Developing a derivation at stage *n* by planning for an unrestricted universal; the parameter a may be any unanalyzed term that is new to the derivation.

We try to reach our goal by a general argument, so we choose as our parameter an unanalyzed term a that is new to the derivation. An instance of ∀x θx for the term a is the goal of the general argument, and further development of the gap lies on the other side of a veil of ignorance concerning that parameter.

The short derivation shown below illustrates these two rules.



At the initial stage here, there is no vocabulary from which a term may be formed—and UI should be used to introduce new terms only as a last resort—so we apply the planning rule to the universal conclusion. Whenever we apply this rule we must introduce a new term as a parameter so, when the rule is applied a second time at stage 2, a second new term is introduced. There is now plenty of vocabulary for use with the exploitation rule. It would have been possible to exploit the initial premise twice to add ∀y Ray as well as ∀y Rby to the resources, and we might have exploited each of these resources twice as well. But the two uses of the exploitation rule that are shown above are enough to derive the resources that enables us to close the gap.

Glen Helman 06 Nov 2004

7.5.s. Summary

The universal quantifiers and conjunction may both be used to say that each of a group of claims is true. This overlap in function indicates an analogy between these logical constants that can be seen also in the laws of entailment for them. The analogue to a component of a conjunction is an instance of a universal, in which applies the universal's quantified predicate is predicated a term. A universal is rarely equivalent to an actual conjunction of its instances, but for a given referential range **R**, it behaves like a possibly infinite conjunction of instances in a language enriched by adding the IDs of all values in **R**—i.e., it behaves like the conjunction of its instances in an expansion of the language by **R**. When we do not fix the range **R**, a universal ∀x θx is not associated with any definite set of instances, but we still know that its instances θτ are all predications of λx θx; and these two features are reflected in the laws of entailment for universals.

In the case of an unrestricted universal, we can state a principle of universal instantiation, which says that the universal implies each of its instances, and we may use this with the law for lemmas to get a law for this sort of universal as a premise. We can describe the role of an unrestricted universal as a conclusion by using the idea of a general argument, in which an instance of a generalization is established in such a way that we may generalize from it to a universal claim. It is sufficient for an argument to be a general one that the term for which the instance is given not be compound, that it not appear in the premises, and that it not appear in the generalization we wish to conclude. Such a term is parametric or a parameter for the argument. The law for the unrestricted conditional as a conclusion then tells us that we can conclude a universal from given premises when we can conclude an instance of it for a parametric term.

In implementing the laws for universals as conclusions, we flag scope lines by terms that are being used as parameters; such terms can appear only to the right of their scope lines. We plan for an unrestricted universal goal by planning to use the rule Universal Generalization (UG). It directs us to set up a flagged scope line with an instance for the parameter as a new goal. While we introduce new terms when planning for universal conclusions, the rule for exploiting universal resources—Universal Instantiation (UI)—should be used only for terms already appearing in the gap—provided there is at least one such term. The exploitation of universals can never be considered complete, and an available universal resource is always an active resource; but exploitation rules do render universals inactive for particular terms and should be applied only to terms for which the universal remains active.

7.5.x. Exercise questions

1. Give the instances of each of the following for the terms a, b, and c (remembering that you will drop the main quantifier, and only the main one, when giving an instance):
- **a.** ∀x Fx
- **b.** ∀y Fy
- **c.** ∀x Rxa
- **d.** ∀x Saxb
- **e.** ∀x ∀y Rxy
- **f.** ∀x (Fx → Gx)
- **g.** ∀x (Fx → Gd)
- **h.** ∀x (Fx → ∀y Rxy)
- **i.** ∀x (Fx → ∀x Rxx)

2. Use the system of derivations to establish each of the following. You may use detachment and attachment rules.
- **a.** ∀x Fx, ∀x (Fx → Gx) ⇒ Ga
- **b.** ∀x (Fx ∧ Gx) ⇒ Fa ∧ Gb
- **c.** ∀x Rxa, ∀x (Rbx → Gx) ⇒ Ga
- **d.** ∀x Fx, ∀x (Fx → Gx) ⇒ ∀x Gx
- **e.** ∀x (Fx ∧ Gx) ⇔ ∀x Fx ∧ ∀x Gx
- **f.** ∀x ∀y Rxy ⇒ (Rab ∧ Rbb) ∧ Rca
- **g.** ∀x ∀y Rxy ⇒ ∀y Rya
- **h.** ∀x ∀y (Rxy → ¬ Ryx) ⇒ ∀x ¬ Rxx
- **i.** ∀x ∀y ∀z ((Rxy ∧ Ryz) → Rxz), ∀x ¬ Rxx ⇒ ∀x ∀y (Rxy → ¬ Ryx)

7.5.xa. Exercise answers

1.

		instance for a	*instance for* b	*instance for* c
a.	∀x Fx	Fa	Fb	Fc
b.	∀y Fy	Fa	Fb	Fc
c.	∀x Rxa	Raa	Rba	Rca
d.	∀x Saxb	Saab	Sabb	Sacb
e.	∀x ∀y Rxy	∀y Ray	∀y Rby	∀y Rcy
f.	∀x (Fx → Gx)	Fa → Ga	Fb → Gb	Fc → Gc
g.	∀x (Fx → Gd)	Fa → Gd	Fb → Gd	Fc → Gd
h.	∀x (Fx → ∀y Rxy)	Fa → ∀y Ray	Fb → ∀y Rby	Fc → ∀y Rcy
i.	∀x (Fx → ∀x Rxx)	Fa → ∀x Rxx	Fb → ∀x Rxx	Fc → ∀x Rxx

2. a.

	∀x Fx	a:1
	∀x (Fx → Gx)	a:2
1 UI	Fa	(3)
2 UI	Fa → Ga	3
3 MPP	Ga	(4)
	•	
4 QED	Ga	

b.

	∀x (Fx ∧ Gx)	a:1, b:3
1 UI	Fa ∧ Ga	2
2 Ext	Fa	(5)
2 Ext	Ga	
3 UI	Fb ∧ Gb	4
4 Ext	Fb	
4 Ext	Gb	(5)
5 Adj	Fa ∧ Gb	(6)
	•	
6 QED	Fa ∧ Gb	

c.

	∀x Rxa	b:1
	∀x (Rbx → Gx)	a:2
1 UI	Rba	(3)
2 UI	Rba → Ga	3
3 MPP	Ga	(4)
	•	
4 QED	Ga	

d.

```
          │ ∀x Fx              a:2
          │ ∀x (Fx → Gx)       a:3
          ├──
          │ ⓐ
2 UI      │ │ Fa               (4)
3 UI      │ │ Fa → Ga          4
4 MPP     │ │ Ga               (5)
          │ │ •
          │ ├──
5 QED     │ │ Ga               1
          ├──
1 UG      │ ∀x Gx
```

e.

```
          │ ∀x (Fx ∧ Gx)       a:3,b:7
          ├──
          │ │ ⓐ
3 UI      │ │ │ Fa ∧ Ga        4
4 Ext     │ │ │ Fa
4 Ext     │ │ │ Ga             (5)
          │ │ │ •
          │ │ ├──
5 QED     │ │ │ Fa             2
          │ ├──
2 UG      │ │ ∀x Fx            1
          │ │ ⓑ
7 UI      │ │ │ Fb ∧ Gb        8
8 Ext     │ │ │ Fb
8 Ext     │ │ │ Gb             (9)
          │ │ │ •
          │ │ ├──
9 QED     │ │ │ Gb             6
          │ ├──
6 UG      │ │ ∀x Gx            1
          ├──
1 Cnj     │ ∀x Fx ∧ ∀x Gx
```

```
          │ ∀x Fx ∧ ∀x Gx      1
          ├──
1 Ext     │ ∀x Fx              a:3
1 Ext     │ ∀x Gx              a:4
          │ ⓐ
3 UI      │ │ Fa               (5)
4 UI      │ │ Ga               (5)
5 Adj     │ │ Fa ∧ Ga          X, (6)
          │ │ •
          │ ├──
6 QED     │ │ Fa ∧ Ga          2
          ├──
2 UG      │ ∀x (Fx ∧ Gx)
```

The term a could have been used again as the parameter of the second general argument of the derivation on the left since we require only that a parameter not appear outside its scope line in the gap which is developed by introducing the general argument, and the first general argument is boxed off from the gap that is developed by setting up the second one. But we will not be short of letters to use as parameters, so it will be easier to see that the requirement is satisfied if we use a new parameter for each general argument in a derivation.

f.

```
          │ ∀x ∀y Rxy              a:1, b:3
          ├──
1 UI      │ ∀y Ray                 b:2
2 UI      │ Rab                    (5)
3 UI      │ ∀y Rby                 b:4, a:6
4 UI      │ Rbb                    (5)
5 Adj     │ Rab ∧ Rbb              X, (7)
6 UI      │ Rba                    (7)
7 Adj     │ (Rab ∧ Rbb) ∧ Rba      X, (8)
          │ •
          ├──
8 QED     │ (Rab ∧ Rbb) ∧ Rba
```

g.

```
          │ ∀x ∀y Rxy          b:2
          ├──
          │ ⓑ
2 UI      │ │ ∀y Rby           a:3
3 UI      │ │ Rba              (4)
          │ │ •
          │ ├──
4 QED     │ │ Rba              1
          ├──
1 UG      │ ∀y Rya
```

Notice that the term a cannot be used as the parameter of the general argument in this derivation because it already appears in the gap (specifically, in its goal) when the general argument is introduced.

h.

```
          │ ∀x ∀y (Rxy → ¬ Ryx)        a:3
          ├──
          │ ⓐ
          │ │ │ Raa                     (5), (6)
          │ │ ├──
3 UI      │ │ │ ∀y (Ray → ¬ Rya)        a:4
4 UI      │ │ │ Raa → ¬ Raa             5
5 MPP     │ │ │ ¬ Raa                   (6)
          │ │ │ •
          │ │ ├──
6 Nc      │ │ │ ⊥                       2
          │ ├──
2 RAA     │ │ ¬ Raa                     1
          ├──
1 UG      │ ∀x ¬ Rxx
```

i.

	∀x ∀y ∀z ((Rxy ∧ Ryz) → Rxz)	a:5
	∀x ¬ Rxx	a:10
	ⓐ	
	ⓑ	
	Rab	(8)
	Rba	(8)
5 UI	∀y ∀z ((Ray ∧ Ryz) → Raz)	b:6
6 UI	∀z ((Rab ∧ Rbz) → Raz)	a:7
7 UI	(Rab ∧ Rba) → Raa	9
8 Adj	Rab ∧ Rba	X,(9)
9 MPP	Raa	(11)
10 UI	¬ Raa	(11)
	•	
11 Nc	⊥	4
4 RAA	¬ Rba	3
3 CP	Rab → ¬ Rba	2
2 UG	∀y (Ray → ¬ Rya)	1
1 UG	∀x ∀y (Rxy → ¬ Ryx)	