7.5.1. Conjunction and universal quantification

The truth conditions of generalizations are analogous to those of conjunctions. So, before looking at laws and rules for the universal quantifiers, we will spend some time comparing these operations to conjunction.

Consider the pair of sentences analyzed below.

Every permanent member of the Security Council supported the resolution

(∀x: Mxs) Sxl

Britain, China, France, Russia, and the U. S. supported the resolution

Sbl ∧ Scl ∧ Sfl ∧ Srl ∧ Sul

[M: λxy (x *is a member of* y); S: λxy (x *supported* y); b: *Britain*; c: *China*; f: *France*; l: *the resolution*; r: *Russia*; s: *the Security Council*; u: *the U. S.*]

These two sentences have the same truth value, but they are not equivalent because in a different possible world the membership of the Security Council could be different. However, consider the sentence

Each of Britain, China, France, Russia, and the U. S. supported the resolution

This could be analyzed in the same way as the second sentence above, but it could be analyzed also as a restricted universal whose restricting predicate is λx (x *is Britain, China, France, Russia, or the U. S.*)—switching to *or* here for the same reasons that lead to us switch in handling *all boys and girls* (see 7.3.2). A full analysis would give us the following:

(∀x: x=b ∨ x=c ∨ x=f ∨ x=r ∨ x=u) Sxl

And this universal *is* equivalent to the conjunction because either way we say that the predicate λx (x *supported* l) is true of the reference values of b, c, f, r, and u.

Each of the universals (∀x: ρx) θx and ∀x θx says that the predicate θ is true of each value in the domain over which it generalizes. Only in special cases (like the example just above) will either be equivalent to a conjunction

$$\theta\tau_1 \wedge \theta\tau_2 \wedge \ldots \wedge \theta\tau_n$$

that predicates θ of each of a series of terms. But it can still be enlightening to compare universals to such conjunctions, so we will develop some vocabulary for doing so. In this section, we will do this only for unrestricted universals, turning to the case of restricted universals in 7.6. Let us say that an ***instance for*** a term τ of a universal ∀x θx is a sentence θτ that applies the quantified predicate θ to τ—that is, an instance of a universal ∀x … x … has the form … τ …, the result of putting τ in place of the occurrences of that variable x that are bound to the quantifier ∀x. An instance asserts of a single reference value what the universal asserts of everything in its domain.

If every reference value is the extension of some term, an unrestricted universal ∀x θx will be true if and only if each of its instances θτ is true. This means that it will behave like a conjunction of these instances. But this is not we could work with such a conjunction in place of the universal because, given just one unanalyzed term and one functor, there will be infinitely many compound terms and infinitely many instances of any universal whose quantifier actually binds a variable. For example, given an unanalyzed term a and functor f, the language will contain the terms

a, fa, f(fa), f(f(fa)), …

and a universal ∀x Px will have the instances

Pa, P(fa), P(f(fa)), P(f(f(fa))), …

Although it is possible to make sense of infinite conjunctions as part of a mathematical structure if there is no expectation that it be possible to write them down, our references to conjunctions of all instances will be only a figure of speech used to motivate and guide our treatment.

For an unrestricted universal to behave like a conjunction of its instances, every reference value must be the value of some term. So let us develop the figure of speech further by imagining that the ID of each reference value in a range **R** is added as a further term of our language. We will speak of this operation as ***expansion by*** **R**. If we expand the language by the range **R** of a structure, an unrestricted universal ∀x θx will be true in that structure if and only if all its instances are true.