6.3.1. Logical properties of identity

The logical properties of identity come from two sources. One is the kind of extension we have stipulated for this relation, the pairs of reference values we say it is true of. The other is the requirement that predicates and functors be extensional, that the compounds they form be transparent to the reference values of their component terms. Properties deriving from this second source are equally properties of the operations of predication and functional application; but since they are not properties of any particular predicate or functor, it is easiest to ascribe them, along with properties of the first sort, to the logical constant =. We will turn first to the properties of identity alone.

What do we know when we know that an equation $\tau = \upsilon$ is true? Well, we know that the terms τ and υ have the same reference value; loosely speaking, we know that they name the same thing. (This is loose speech, first, because the terms may not be names but rather definite descriptions and, second, because the reference value of the terms may be nil, in which case neither names anything.) So we might say that τ and υ are each "aliases" of their common reference value. It will be convenient to have a way of speaking of such terms in relation to each other rather than in relation to their value, so let us say that they are aliases in relation to each other—or, more briefly, that they are ***co-aliases***.

This leads us immediately to a property of identity. For the relation of having the same reference value, of being co-aliases, is ***symmetric***. It does not order the two terms in any way; if we can assert it of them taken in one order, we can equally we assert it of them the other way around. This gives us our first principle for =, the ***law of symmetry for identity***:

$$\tau = \upsilon \Rightarrow \upsilon = \tau.$$

This principle is stated as an entailment, but it implies that the two equations are equivalent since it licenses reversals of equations and we can undo a reversal by reversing again.

Now suppose that we know not only that a term υ is an alias for a term τ but also that τ is an alias for a term σ. All three terms must then have the same reference value, so we could say that υ is an alias for σ. Putting this more formally, we have a second principle, the ***law of transitivity for identity***:

$$\sigma = \tau, \tau = \upsilon \Rightarrow \sigma = \upsilon.$$

Again, there is a more symmetric principle lurking in the background—namely, that any two of these three equations entails the third. But this principle is harder to state compactly, and a fuller investigation of it would show that it also relies on the law of symmetry.

The two laws we have stated tell us that certain equations are true *if* others are, but they do not commit us categorically to the truth of any equations at all. How do we know there are any true equations? Well, what would it take for there to be none. Perhaps this would be so if were no aliases in the ordinary sense and every reference value was the extension of at most one term. We do not want to rule this out, for our laws are supposed to be very general and should not make any assumptions about the richness of our non-logical vocabulary. But even in a case like this, if there is any term at all in our language, we can form an equation with this term taken twice and the equation will be true. And that is one way of stating the ***law of reflexivity for identity***:

$$\Rightarrow \tau = \tau.$$

So there will be true equations if there are any equations at all.

We have found three properties of identity that derive from the kind of extension we have stipulated for =. Collecting them, we have:

Reflexivity. $\Rightarrow \tau = \tau$.
Symmetry. $\tau = \upsilon \Rightarrow \upsilon = \tau$.
Transitivity. $\sigma = \tau, \tau = \upsilon \Rightarrow \sigma = \upsilon$.

Identity is not the only predicate that has these properties. For example, the predicate λx (x *has the same shape as* y) obeys analogous laws; and that example should suggest many others. A predicate for which laws of reflexivity, symmetry, and transitivity hold is said to express an ***equivalence relation*** because such a relation attributes sameness in some respect. For example, if we grant that a line is parallel to itself, we can say that λx (x *is parallel to* y) expresses an equivalence relation; and this relation attributes sameness in direction.

An extreme example of an equivalence relation is the relation that holds between any pair of reference values (including any reference value and itself). Since this relation never fails to hold there is no way for it to violate any of the three laws, and it must be an equivalence relation. The extension of the identity predicate is at the other extreme of equivalence relations. If we represent the two in tabular form as truth-valued functions of reference values, we have something like this.

	0	1	2	3	⋯
0	T	T	T	T	⋯
1	T	T	T	T	⋯
2	T	T	T	T	⋯
3	T	T	T	T	⋯
⋮	⋮	⋮	⋮	⋮	⋱

=	0	1	2	3	⋯
0	T	F	F	F	⋯
1	F	T	F	F	⋯
2	F	F	T	F	⋯
3	F	F	F	T	
⋮	⋮	⋮	⋮		⋱

The first relation has **T** everywhere while the extension of = has **T** only along the diagonal from the upper left to the lower right. Identity holds in the fewest cases possible for an equivalence relation because, if any of the pairs along the diagonal were dropped, the law of reflexivity would not hold. Since identity thus expresses the narrowest equivalence relation, we might think of it as expressing sameness in all respects.

Although the status of identity as the narrowest equivalence relation derives from the extension we have stipulated for =, this does not provide a property that we can express in laws for = alone. Our ability to express the idea of sameness in all respects depends on the predicates and functors we have available to express a variety of "respects." What we can say is that identity implies sameness with regard to each predicate and functor, and we can find further properties of identity by exploiting the consequences of idea. First consider a one-place predicate —say λx (x *is red*). Two things are the same with respect to redness if both are red or neither is. Hence, to say that identity implies sameness with respect to this predicate is say that an equation $\tau = \upsilon$ implies that τ *is red* and υ *is red* have the same truth value. We have no very good way of expressing this sort of relation among three sentences directly, but the symmetry of = means that it is enough to say that $\tau = \upsilon$ and τ *is red* together entail υ *is red*. Generalizing this to any one-place predicate F leads us to assert the law

$$\tau = \upsilon, F\tau \Rightarrow F\upsilon.$$

This is as at least part of what is involved in saying that identity implies sameness in all respects. In fact, if we put θ in place of F and thus allow the predicate to be an abstract, this law says it all. But, for the moment, we will consider only predicates that are not abstracts and say that an equivalence relation that supports a law of this form for a given predicate F is a ***congruence for*** F. An equivalence relation that implies sameness with respect to redness (e.g., the extension of λx (x *has the same color as* y)) is thus a congruence for λx (x *is red*). (The source of the term *congruence* is the geometrical relation of congruence, which implies sameness with respect to size and shape though not with respect to location.)

The form of this law ought to suggest something that is familiar from elementary algebra, the use of an equation to replace one expression by another. Now, in algebra we can equally well use more than one equation to make several replacements simultaneously, and congruence principles can take a similar form. Consider sameness with respect to the relation expressed by a 2-place predicate such as λx (x *is younger than* y). Things that are the same in this respect should be younger than the same things and have the same things younger than them. We can express this idea compactly by the following:

$$\tau_1 = \upsilon_1, \tau_2 = \upsilon_2, \tau_1 \textit{ is younger than } \tau_2 \Rightarrow \upsilon_1 \textit{ is younger than } \upsilon_2$$

And we can claim this holds for 2-place predicates generally by stating the law

$$\tau_1 = \upsilon_1, \tau_2 = \upsilon_2, R\tau_1\tau_2 \Rightarrow R\upsilon_1\upsilon_2.$$

In these statements, we have economized by speaking of both places of the predicate in a single law. Since τ_1 and υ_1 could be the same term and so could τ_2 and υ_2, the law covers cases where a change is made in only one of the two places of R. An equivalence relation that supports a law like this one for identity is said to be a ***congruence for*** the predicate R. The relation of having the same age will be a congruence in this sense for the relation expressed by λx (x *is younger than* y).

Now it should be clear that we might state a law like these two that applies to identity and a predicate P with any number of places:

Congruence for P. $\tau_1 = \upsilon_1, \ldots, \tau_n = \upsilon_n, P\tau_1\ldots\tau_n \Rightarrow P\upsilon_1\ldots\upsilon_n$ (where P has *n* places).

A large part of what we mean by saying that identity implies sameness in all respects can be captured by saying that it is a congruence for all predicates.

A large part, but not all. We have not yet said anything about functors. Here we can make the story short because the law we want is more familiar. It is this:

Congruence for f. $\tau_1 = \upsilon_1, \ldots, \tau_n = \upsilon_n \Rightarrow f\tau_1\ldots\tau_n = f\upsilon_1\ldots\upsilon_n$ (where f has *n* places).

This says that an equation between compound terms $f\tau_1\ldots\tau_n$ and $f\upsilon_1\ldots\upsilon_n$ follows from equations between their corresponding components. We can have laws like this for equivalence relations besides identity; and, when we have such a law for an equivalence relation, the relation is said to be a ***congruence for*** the functor f. The relation of having the same absolute value (i.e., of being equal or differing only in sign) is a congruence for a functor expressing the squaring function (or the cosine function). In the case of identity, we can claim congruence for *all* functors.

Have we now captured the properties of identity by saying that it is a congruence for all predicates and all functors? The laws we have stated suffice to capture all true general principles of entailment involving identity, and that was our aim. We might still ask whether a relation could obey these laws without being a relation of sameness in all respects. The question comes to something like this: are the features of a thing that are expressible by predicates and functors sufficient to pin down its identity, to distinguish it from all other things? This is a puzzling question. While any given collection of predicates and functors can certainly fail to express differences among things, it is hard to pin down the claim that there could be such differences that are expressible by no predicates or functors whatsoever, for any attempt to say what such differences might be would begin to undercut the claim that they are inexpressible. In any case, asserting the laws above for all predicates and functors suffices to establish all general principles of entailment concerning identity that we can express using our analysis of logical form.

In saying that identity is a congruence for predicates and functors, we say that predicates and functors are extensional operations and, in particular, that they form referentially transparent compounds. For example, if we were to count the sentence-with-a-gap *For the past two centuries, ___ has been over 35* as a predicate, we could not say that identity is a congruence for all predicates because to assert congruence for this incomplete expression would be to assert the validity of the argument

> *For the past two centuries, the U. S. president has been over 35*
> *The U. S. president* = *George Bush*
> ____
> *For the past two centuries, George Bush has been over 35*

and, as was noted in 6.1.2, this is naturally understood to have true premises and a false conclusion.

This raises a wider philosophical and logical issue. Could we at least say that this sentence-with-a-blank has an extension that is a function? Such a function would have to yield truth values as output based on something beyond the reference values of the terms to which it was applied, and we might speak of it as an ***intensional property*** (as distinct from as a ***property in intension***, which is merely the way the extensional property expressed by an ordinary predicate varies from world to world). So one part of the question we have just asked is whether there are intensional properties.

The other part is whether there is anything for an intensional property to be a property of. It cannot be a property of an object thought of as a reference value because it depends on distinctions that are ignored in saying what reference value a term has. One way of putting this side of the issue is to ask whether there is any sense of *thing* in which the terms *the U. S. president* and *George Bush* could be said to signify different things. Perhaps we could say that one signifies a public official and the other signifies a person and say that one and the same public official could be identical with different people at different times. The oddity of this talk suggests that nasty problems might lurk here, so we will not open this door any wider. Suffice it to say that logicians and philosophers have adopted a full range of positions on this issue. Some happily accept ***intensional entities*** (such as public officials as distinct from the people who hold those offices) while others reject all talk of intensions, not only of intensional entities and intensional properties but even of the intensions of ordinary extensional predicates.

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