4.2. Arguing from and for alternatives

4.2.0. Overview

Because a disjunction normally says less than its components while a conjunction says more, the two connectives play very different roles in entailment.

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4.2.1. Proofs by cases

The validity of the argument

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam either praised or condemned the proposal
———
Sam granted the proposal's significance.

can be accounted for by the validity of the following two arguments:

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam praised the proposal
———
Sam granted the proposal's significance

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam condemned the proposal
———
Sam granted the proposal's significance.

Each replaces the disjunctive third premise of the original argument by one of its two components. This way of establishing an entailment is sometimes called a ***proof by cases***. In this example, the two cases are Sam having praised the proposal and Sam having condemned it. Since the disjunction says all and only what is common to these two alternatives, something follows from the disjunction in isolation or in addition to other premises just in case it follows from each one of these alternatives under similar circumstances.

More formally, the idea behind proofs by cases is captured by a ***law for disjunction as a premise***:

$\Gamma, \phi \vee \psi \Rightarrow \chi$ if and only if both $\Gamma, \phi \Rightarrow \chi$ and $\Gamma, \psi \Rightarrow \chi$

To see why this law is true note that to divide the members of Γ and $\phi \vee \psi$ on the one hand from χ on the other, a possible world must make $\phi \vee \psi$ and all members of Γ true while making χ false. To do this it must make at least one of ϕ and ψ true, and that means that it must divide at least one of the arguments Γ, ϕ / χ and Γ, ψ / χ. So, to say that the original argument is valid is to say that neither of these latter arguments can have its premises and alternatives divided—that is, that both are valid.

This idea appears in derivations by way of a rule we will call ***Proof by Cases*** (PC); it is shown in Figure 4.2.1-1.

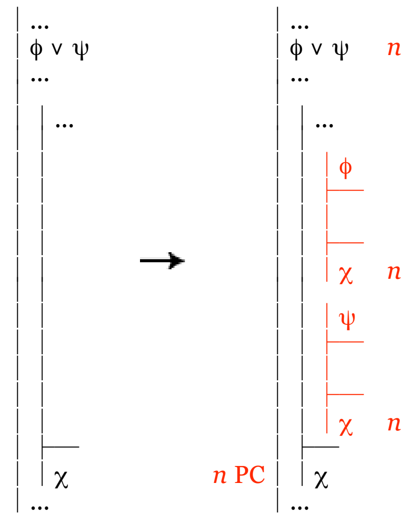
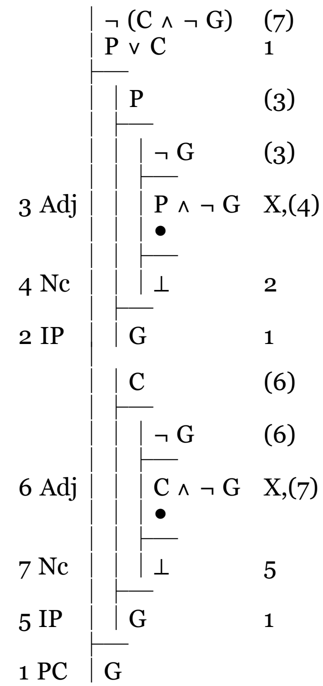


Fig. 4.2.1-1. Developing a derivation by exploiting a disjunction at stage *n*.

PC divides a gap into two new gaps. Each is a ***case argument*** that retains the original goal but adds one of the components of the disjunction as a supposition. The function of each supposition is to specify one of the two alternative cases in which the original disjunction is true. A supposition is required because, although our premises tell us that at least one of the disjuncts is true, we do not know which that is, and the one that is true will, in general, vary among the possible worlds in which the resources of the original gap are true.

Here is a derivation which uses this rule to provide a proof for example with which we began.



[C: *Sam condemned the proposal*; G: *Sam granted the proposal's significance*; P: *Sam praised the proposal*]

In the two case arguments, we suppose first that Sam praised the proposal and then that he condemned it and, in each case, we show that he granted the proposal's significance (by showing that he could not have failed to grant it). Since at least one of these two cases must be true whenever the premises are all true, we know that the conclusion must be true also.

4.2.2. Proving disjunctions

Now let us look at disjunctions as conclusions. An entailment $\Gamma \Rightarrow \phi \vee \psi$ will hold if and only if $\phi \vee \psi$ is true in every possible world in which all members of Γ are true. But this is to say that at least one of ϕ and ψ is true in every such world, and that is a way of saying that Γ renders ϕ and ψ jointly exhaustive. So we can state the following principle:

$\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma \Rightarrow \phi, \psi$

Since the right-hand side has two alternatives, this is not a law concerning entailment alone, and we will not take the principle in this form as our account of the role of disjunctions as conclusions. However, we can use the basic law for relative exhaustiveness to restate the right-hand side as claim of entailment.

Indeed we have two ways of doing that. If ϕ and ϕ' are contradictory, we can say

$\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \phi' \Rightarrow \psi$

and if ψ and ψ' are contradictory, we can say

$\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \psi' \Rightarrow \phi$

In short, a disjunction is a valid conclusion from premises Γ if and only if adding to our premises a sentence contradictory to one disjunct enables us to validly conclude the other disjunct.

In stating a principle for disjunction we will limit ourselves to cases where a sentence and its negation are the pair of contradictory sentences. But, when the disjuncts are already negative, that leaves us with two choices for each of the pairs ϕ and ϕ' and ψ and ψ' since each of ϕ' and ψ' might be the result of either adding or dropping a negation. To avoid stating four principles to cover each of these possibilities, we will introduce some notation to capture the general idea of obtaining a contradictory sentence by either adding or dropping a negation. Let the notation $\overline{\phi}$ (read *ϕ bar*) stand for $\neg\,\phi$ when ϕ is not a negation and, when ϕ is the negation $\neg\,\chi$, for either $\neg\,\neg\,\chi$ or χ. That is, $\overline{\phi}$ is the result of either negating or, if possible, de-negating ϕ. We will say that $\overline{\phi}$ ***bars*** ϕ or is the ***barring*** of ϕ.

Then $\overline{\phi}$ and ϕ form a contradictory pair consisting of a sentence and its negation in one order or the other, so we may formulate our ***law for disjunction as a conclusion*** with only two statements:

(i) $\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \overline{\phi} \Rightarrow \psi$, and
(ii) $\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \overline{\psi} \Rightarrow \phi$

When these are implemented as derivation rules, they give us two ways of planning for a disjunctive goal. The two rules are shown as alternative developments in Figure 4.2.2-1. We will refer to both forms of the rule as ***Proof of Exhaustion*** (PE) since it is a way of showing that ϕ and ψ, taken together, exhaust all possibilities left open by the premises.

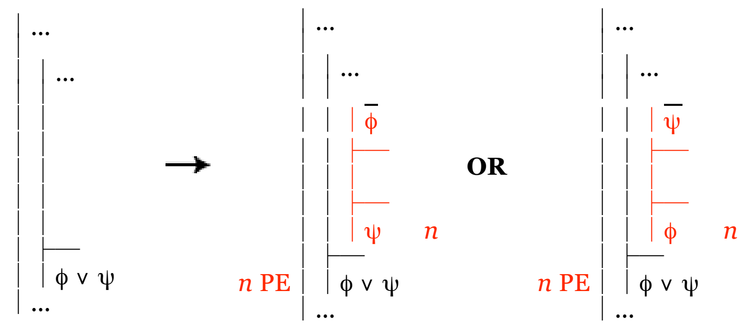


Fig. 4.2.2-1. Alternative ways of developing a derivation by planning for a disjunction at stage *n*.

In each way of developing a gap, we set one of the components of the disjunction as a new goal and add the barring (i.e., negation or de-negation) of the other component as a supposition. Both forms of planning will lead to the same answer in the end, but one or the other may be more efficient in a particular case. There is no simple way of predicting which choice is best but the following rules of thumb may help:

(i) if only one component is a negation, choose it to form the supposition (by dropping its negation);
(ii) if only one component is a non-negative compound choose it as the goal;
(iii) if only one component seems likely to figure in closing the gap and it is not a negation, choose it as the goal.

In many cases none of these suggestions will apply; but, in most such cases, neither one of the two forms of the rule is better than the other.

The supposition in PE may be described as ***hypothetical***, and this indicates a third role that suppositions may play. In *reductio* arguments and indirect proofs, we make suppositions with the aim of showing that they are false. In a proof by cases we make a pair of suppositions at least one of which we take to be true. In PE on the other hand, a supposition is made with no expectation of either truth or falsity. It is made instead simply to establish a connection between it and another claim. As we argue within the scope of the supposition, we are making a ***hypothetical argument***, one that explores the implications of the supposition in order to establish a connection between it and another claim. The conclusion we draw to end the scope of the supposition states this connection between the two claims. Here, it is $\phi \vee \psi$, so the connection between the two sentences is at least one of them is true. This is a statement made categorically; this, it no longer falls under the supposition.

There is some danger of getting tangled in the terminology here, so let's pause and look at it more closely. The terms *hypothetical* and *categorical* derive from an ancient classification of sentences into the "categorical," the "disjunctive," and the "hypothetical". Since disjunctions and "hypothetical sentences" (the conditionals to be studied in the next chapter) are ways of hedging claims, the term ***categorical*** has acquired the meaning 'unhedged'. Now the disjunctive goal to which we applied this term above certainly hedges each of its components, so it does not state them categorically. But, while sentences along the scope line of the hypothetical argument are stated only "under the hypothesis" that is the supposition of this argument, the disjunction following the argument is no longer hedged in this way, which means that it is stated categorically with respect to that supposition (though it may still fall in the scope of earlier ones). In short, when the scope line of a hypothetical argument ends, a hedged statement (of a possibly unhedged sentence) is converted into an unhedged statement of a sentence that incorporates a hedge.

As an example of this rule, consider the argument below, understanding *X was out* to be the denial of *X was home*. The validity of this argument can be established by the English derivation whose first stage is shown at the right.

Ann and Bill were not both home without the car being in the driveway
The car was not in the driveway
———
Either Ann or Bill was out

```
       │ ¬ ((A ∧ B) ∧ ¬ C)
       │ ¬ C
       ├──
       │ │ A
       │ ├──
       │ │
       │ ├──
       │ │ ¬ B                 1
       ├──
  1 PE │ ¬ A ∨ ¬ B
```

The overall form is that of a hypothetical argument in which we suppose that Ann was at home (a supposition that is one of the two possibilities for ¬ A) and establish under this hypothesis that Bill was out. This shows the connection between Ann being out and Bill being out that we claim when we state categorically that at least one was out. When the hypothetical argument ends, we move from a statement of ¬ B under the hypothesis A to a statement of ¬ A ∨ ¬ B that is hedged by the added alternative ¬ A but that is no longer stated under the hypothesis A.

Notice that if we continue the derivation

```
        │ ¬ ((A ∧ B) ∧ ¬ C)
        │ ¬ C
        ├──
        │ │ A
        │ ├──
        │ │ │ B
        │ │ ├──
        │ │ │
        │ │ ├──
        │ │ │ ⊥                 2
        │ │ ├──
  2 RAA │ │ ¬ B                 1
        ├──
   1 PE │ ¬ A ∨ ¬ B
```

we plan for the goal ¬ B by supposing B for *reductio*. And this example illustrates the different functions of the two sorts of supposition. We suppose that Ann is home in order to show that ¬ B (*Bill is out*) is true in all possible worlds in which ¬ A (*Ann is out*) is false. We go on to show that ¬ B is true in these cases by showing that to suppose further that B would rule out all possibilities—i.e., that this supposition would be absurd when added to our premises and the supposition A. From one point of view, both suppositions are merely added assumptions. But we add the first in order to show that, by adding the second, we would go too far.

4.2.3. Further examples

Both disjunction rules are illustrated by the derivation at the right, in which one grouping of a three-part disjunction is shown to entail the other. Choices between the two ways of planning for a goal disjunction were made at stages 2, 3, 5, 6, and 7 in accordance with the rules of thumb given earlier. The way each choice was made helped to shorten the derivation —though in each case only by a few steps. This derivation is contrived to provide several examples of PE; instead we might have planned for the initial goal at stage 1 before exploiting the premise rather than planning for it separately in each of three gaps.

```
        | A ∨ (B ∨ C)                1
        |—
        | | A                        (4)
        | |—
        | | | ¬ C
        | | |—
        | | | | ¬ B
        | | | |—
        | | | | •
        | | | |—
4 QED   | | | | A                    3
        | | |—
3 PE    | | | A ∨ B                  2
        | |—
2 PE    | | (A ∨ B) ∨ C              1
        | | B ∨ C                    5
        | |—
        | | | B                      (8)
        | | |—
        | | | | ¬ C
        | | | |—
        | | | | | ¬ A
        | | | | |—
        | | | | | •
        | | | | |—
8 QED   | | | | | B                  7
        | | | |—
7 PE    | | | | A ∨ B                6
        | | |—
6 PE    | | | (A ∨ B) ∨ C            5
        | | | C                      (10)
        | | |—
        | | | | ¬ (A ∨ B)
        | | | |—
        | | | | •
        | | | |—
10 QED  | | | | C                    9
        | | |—
9 PE    | | | (A ∨ B) ∨ C            5
        | |—
5 PC    | | (A ∨ B) ∨ C              1
        |—
1 PC    | (A ∨ B) ∨ C
```

The scale of the difference you can expect to result from a choice between the two forms of PE is illustrated by the two derivations below.

```
        | B                  (3)
        |—
        | | ¬ A
        | |—
        | | | ¬ C
        | | |—
        | | | •
        | | |—
3 QED   | | | B              2
        | |—
2 PE    | | B ∨ C            1
        |—
1 PE    | A ∨ (B ∨ C)
```

```
        | B                      (5)
        |—
        | | ¬ (B ∨ C)            3
        | |—
        | | | ¬ A
        | | |—
        | | | | | ¬ C
        | | | | |—
        | | | | | •
        | | | | |—
5 QED   | | | | | B              4
        | | | |—
4 PE    | | | | B ∨ C          3
        | | |—
3 CR    | | | ⊥                2
        | |—
2 IP    | | A                  1
        |—
1 PE    | A ∨ (B ∨ C)
```

Each chooses a different way of planning for the initial goal at stage 1. Notice that in the second, which makes the less efficient choice, we are led back to the goal B ∨ C in a couple of stages.

4.2.4. The duality of conjunction and disjunction

While a conjunction and a disjunction formed from the same components are certainly not contradictories, the two connective are opposites in another sense, the one for which we have used the term *dual*.

This duality can be expressed in one way by saying that when conjunction and disjunction are applied to pairs of sentences whose corresponding components are contradictory, the results are contradictory. As an example, let us again take *X was home* and *X was out* to be contradictories. Now to get a sentence contradictory to *Ann and Bill were home*, we cannot take *Ann and Bill were out* since both sentences would be false if one of Ann and Bill was home and the other out. To get a contradictory to *Ann and Bill were home* we need to leave open those possibilities, and *Ann or Bill was out* will do this. That is, *Ann and Bill were home* is contradictory to *Ann or Bill was out* and, similarly, *Ann or Bill was home* is contradictory to *Ann and Bill were out*. And this is to say that ¬ *Ann and Bill were home* ⇔ *Ann or Bill was out* and that ¬ *Ann or Bill was home* ⇔ *Ann and Bill were out*.

When they are limited to the cases of contradictoriness captured by the bar notation, these patterns of equivalence are know as ***De Morgan's laws***:

$$\neg\,(\phi \wedge \psi) \Leftrightarrow \overline{\phi} \vee \overline{\psi}$$
$$\neg\,(\phi \vee \psi) \Leftrightarrow \overline{\phi} \wedge \overline{\psi}$$

Although these are named after Augustus De Morgan (1806-1871), they were known long before his time.

Another way to see the duality of conjunction and disjunction is to look at the principles that hold for them with respect to relative exhaustiveness. The table below follows the pattern of the one given for ⊤ and ⊥ in 1.4.6.

	as a premise	as an alternative
Conjunction	$\Gamma, \phi \wedge \psi \Rightarrow \Delta$ iff $\Gamma, \phi, \psi \Rightarrow \Delta$	$\Gamma \Rightarrow \phi \wedge \psi, \Delta$ iff both $\Gamma \Rightarrow \phi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$
Disjunction	$\Gamma, \phi \vee \psi \Rightarrow \Delta$ iff both $\Gamma, \phi \Rightarrow \Delta$ and $\Gamma, \psi \Rightarrow \Delta$	$\Gamma \Rightarrow \phi \vee \psi, \Delta$ iff $\Gamma \Rightarrow \phi, \psi, \Delta$

(Here *iff* is used as an abbreviation of *if and only if*.) Notice that the analogy between the upper left and lower right and between the lower left and upper right. That is, conjunction behaves as a premise in much the way disjunction behaves as an alternative and disjunction behaves as premise in much the way conjunction behaves as an alternative.

Since ⊤ and ⊥ are paired as duals and so are conjunction and disjunction, you might wonder what serves as the dual to negation. In fact, it is dual to itself. If we negate each of a pair of contradictory sentences, the results are contradictory; that is, we do not need to apply different operations to the two contradictory sentences in order for the results to be contradictory. Notice also that the behavior of negation as a premise is analogous to its behavior as an alternative.

$$\Gamma, \neg\,\phi \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \phi, \Delta$$
$$\Gamma \Rightarrow \neg\,\phi, \Delta \text{ iff } \Gamma, \phi \Rightarrow \Delta$$

Having a negated premise or alternative is equivalent to having the unnegated sentence in the opposite role.

As was noted in 1.4.6, the term *duality* points to a certain sort of two-for-one principle. In particular, it is used when there is some way of associating vocabulary items as pairs so that replacing one member of a pair by the other throughout any truth will yield another truth. In our case, we have the associations

⇒ —— ⇐
⇔ —— ⇔
⊤ —— ⊥
¬ —— ¬
∧ —— ∨

The equivalence arrow is dual to itself because it amounts to having both ⇒ and ⇐; and if each of them is reversed, we still have both.

Let us apply this association in a couple of examples. The principle

$$\Gamma, \phi \wedge \psi \Rightarrow \Delta \text{ iff } \Gamma, \phi, \psi \Rightarrow \Delta$$

(the principle on the upper left of the table above) turns into

$$\Gamma, \phi \vee \psi \Leftarrow \Delta \text{ iff } \Gamma, \phi, \psi \Leftarrow \Delta$$

which (once it is rewritten with premises on the left and alternatives on the right and the premises and alternatives re-ordered) amounts to

$$\Delta \Rightarrow \phi \vee \psi, \Gamma \text{ iff } \Delta \Rightarrow \phi, \psi, \Gamma$$

This last statement is the principle of the lower right of the table with the variables Γ and Δ interchanged. And the principle

$$\Gamma, \neg\,\phi \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \phi, \Delta$$

becomes

$$\Gamma, \neg \phi \Leftarrow \Delta \text{ iff } \Gamma \Leftarrow \phi, \Delta$$

or

$$\Delta \Rightarrow \neg \phi, \Gamma \text{ iff } \Delta, \phi \Rightarrow \Gamma$$

which is the second of the principles for negation stated above with Γ and Δ interchanged. In the next section, we will see more examples of such transformations between principles on the basis of dual concepts, but we have already seen other examples: each of the two forms of De Morgan's laws may be transformed into the other by this sort of association.

Glen Helman 27 Sep 2004

4.2.s. Summary

A disjunction $\phi \vee \psi$ is false only when its disjuncts are both false, and it thus says only what is their shared content. The law for disjunction as a premise tell us that we can establish a conclusion using such a premise by showing that it is entailed by each of the disjuncts (given our other premises). This way of exploiting a disjunction is known as a proof by cases, and it appears in our system of derivations as a rule Proof by Cases (PC) that leads us to divide a gap into two case arguments, each of which takes over the original goal and adds one of the two disjuncts as a supposition.

To show that a disjunction is a valid conclusion, we must show that its disjuncts are rendered jointly exhaustive by the premises. We can do this by showing that one of the disjuncts will follow if we add the contradictory of the other to our premises. In order conveniently refer to a contradictory obtained by either negating or de-negating a sentence, we use the bar notation to indicate a sentence $\bar{\phi}$ that either is the negation of ϕ or has ϕ as its negation. The law for disjunction as a conclusion then tells us that we can conclude a disjunction if we can conclude one disjunct provided we take the barring of the other disjunct as a premise. The rule implementing this idea is Proof of Exhaustion. It enables us to conclude a disjunction from an argument that may be called hypothetical since it draws a conclusion that we may not be prepared to assert categorically by arguing under a supposition in order to establish a relation between the two claims. It does not matter for the soundness or safety of PE which disjunct figures as the goal of this hypothetical argument and which is barred in its supposition.

Derivations, especially those that have a disjunction as a goal as well as a premise can often be developed in a number of different ways. Some of these can be significantly longer than others but the choice between forms of PE will usually have only a limited impact on the length.

Conjunction and disjunction are opposite in the sense of being dual. One manifestation of this relation is in De Morgan's laws, which tell how to restate the denial of a conjunction or disjunction as an assertion of the other form of compound. Another manifestation is a pattern in laws of relative exhaustiveness which allows us to interchange conjunctions and disjunctions if at the same time we interchange ⊤ and ⊥ and also premises and alternatives.

Glen Helman 28 Sep 2004

4.2.x. Exercises

1. Use derivations to establish each of the claims of entailment and equivalence shown below. (Remember that claims of equivalence require derivations in both directions.)

- **a.** A ∧ B ⇒ A ∨ B
- **b.** A ∧ B ⇒ B ∨ C
- **c.** A ∨ B, ¬ A ⇒ B
- **d.** A ∨ (A ∧ B) ⇒ A
- **e.** A ∨ B, ¬ (A ∧ C), ¬ (B ∧ C) ⇒ ¬ C
- **f.** A ∧ (B ∨ C) ⇒ (A ∧ B) ∨ C
- **g.** A ∨ B, C ⇒ (A ∧ C) ∨ (B ∧ C)
- **h.** A ∨ B, ¬ A ∨ C ⇒ B ∨ C
- **i.** A ⇔ (A ∧ B) ∨ (A ∧ ¬ B)

2. Use derivations to establish each of the claims of equivalence below.

- **a.** A ∨ A ⇔ A
- **b.** A ∨ B ⇔ B ∨ A
- **c.** A ∨ (B ∨ C) ⇔ (A ∨ B) ∨ C
- **d.** A ∨ (B ∧ ¬ B) ⇔ A
- **e.** ¬ (A ∨ B) ⇔ ¬ A ∧ ¬ B
- **f.** ¬ (A ∧ B) ⇔ ¬ A ∨ ¬ B

3. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample that divides an open gap.

- **a.** A ∨ B, A ⇒ ¬ B
- **b.** A ∨ (B ∧ C) ⇔ (A ∨ B) ∧ C
- **c.** ¬ (A ∨ B) ⇔ ¬ A ∨ ¬ B

4.2.xa. Exercise answers

1. a.

```
          │ A ∧ B    1
          ├──
1 Ext     │ A
1 Ext     │ B        (3)
          │
          │ │ ¬ A
          │ ├──
          │ │ •
          │ ├──
3 QED     │ │ B      2
          ├──
2 PE      │ A ∨ B
```

b.

```
          │ A ∧ B    1
          ├──
1 Ext     │ A
1 Ext     │ B        (3)
          │
          │ │ ¬ C
          │ ├──
          │ │ •
          │ ├──
3 QED     │ │ B      2
          ├──
2 PE      │ B ∨ C
```

c.

```
          │ A ∨ B        1
          │ ¬ A          (3)
          ├──
          │ │ A          (3)
          │ ├──
          │ │ │ ¬ B
          │ │ ├──
          │ │ │ •
          │ │ ├──
3 Nc      │ │ │ ⊥        2
          │ ├──
2 IP      │ │ B          1
          │
          │ │ B          (4)
          │ ├──
          │ │ •
          │ ├──
4 QED     │ │ B          1
          ├──
1 PC      │ B
```

d.

```
        | A ∨ (A ∧ B)      1
        |----
        | | A              (2)
        | |----
        | | •
        | |----
2 QED   | | A              1
        | | A ∧ B          3
        | |----
3 Ext   | | A              (4)
3 Ext   | | B
        | | •
        | |----
4 QED   | | A              1
        |----
1 PC    | A
```

e.

```
          | A ∨ B                 2
          | ¬ (A ∧ C)             3
          | ¬ (B ∧ C)             7
          |----
          | | C                   (6),(10)
          | |----
          | | | A                 (5)
          | | |----
          | | | | | •
          | | | | |----
5 QED     | | | | | A             4
          | | | | | •
          | | | | |----
6 QED     | | | | | C             4
          | | | |----
4 Cnj     | | | | A ∧ C         3
          | | |----
3 CR      | | | ⊥                 2
          | | | B                 (9)
          | | |----
          | | | | | •
          | | | | |----
9 QED     | | | | | B             8
          | | | | | •
          | | | | |----
10 QED    | | | | | C             8
          | | | |----
8 Cnj     | | | | B ∧ C         7
          | | |----
7 CR      | | | ⊥                 2
          | |----
2 PC      | | ⊥                   1
          |----
1 RAA     | ¬ C
```

f.

```
        | A ∧ (B ∨ C)          1
        |----
1 Ext   | A                    (5)
1 Ext   | B ∨ C                2
        | | B                  (6)
        | |----
        | | | ¬ C
        | | |----
        | | | | •
        | | | |----
5 QED   | | | | A            4
        | | | | •
        | | | |----
6 QED   | | | | B            4
        | | |----
4 Cnj   | | | A ∧ B          3
        | |----
3 PE    | | (A ∧ B) ∨ C      2
        | | C                  (8)
        | |----
        | | | ¬ (A ∧ B)
        | | |----
        | | | •
        | | |----
8 QED   | | | C              7
        | |----
7 PE    | | (A ∧ B) ∨ C      2
        |----
2 PC    | (A ∧ B) ∨ C
```

g.

```
        | A ∨ B                      1
        | C                          (5),(9)
        |----
        | | A                        (4)
        | |----
        | | | ¬ (B ∧ C)
        | | |----
        | | | | •
        | | | |----
4 QED   | | | | A                  3
        | | | | •
        | | | |----
5 QED   | | | | C                  3
        | | |----
3 Cnj   | | | A ∧ C                2
        | |----
2 PE    | | (A ∧ C) ∨ (B ∧ C)      1
        | | B                        (8)
        | |----
        | | | ¬ (A ∧ C)
        | | |----
        | | | | •
        | | | |----
8 QED   | | | | B                  7
        | | | | •
        | | | |----
9 QED   | | | | C                  7
        | | |----
7 Cnj   | | | B ∧ C                6
        | |----
6 PE    | | (A ∧ C) ∨ (B ∧ C)      1
        |----
1 PC    | (A ∧ C) ∨ (B ∧ C)
```

h.

```
         │ A ∨ B                 1
         │ ¬ A ∨ C               2
         ├──
         │ │ A                   (5)
         │ ├──
         │ │ │ ¬ A               (5)
         │ │ ├──
         │ │ │ │ ¬ B
         │ │ │ ├──
         │ │ │ │ │ ¬ C
         │ │ │ │ ├──
         │ │ │ │ │ •
         │ │ │ │ ├──
5 Nc     │ │ │ │ │ ⊥           4
         │ │ │ ├──
4 IP     │ │ │ │ C             3
         │ │ ├──
3 PE     │ │ │ B ∨ C           2
         │ │ │ C               (7)
         │ │ ├──
         │ │ │ │ ¬ B
         │ │ │ ├──
         │ │ │ │ •
         │ │ │ ├──
7 QED    │ │ │ │ C             6
         │ │ ├──
6 PE     │ │ │ B ∨ C           2
         │ ├──
2 PC     │ │ B ∨ C             1
         │ │ B                 (9)
         │ ├──
         │ │ │ ¬ C
         │ │ ├──
         │ │ │ •
         │ │ ├──
9 QED    │ │ │ B               8
         │ ├──
8 PE     │ │ B ∨ C             1
         ├──
1 PC     │ B ∨ C
```

i.

```
         │ A                         (3),(7)
         ├──
         │ │ ¬ (A ∧ B)               5
         │ ├──
         │ │ │ •
         │ │ ├──
3 QED    │ │ │ A                     2
         │ │ │ │ B                   (8)
         │ │ │ ├──
         │ │ │ │ │ │ •
         │ │ │ │ │ ├──
7 QED    │ │ │ │ │ │ A               6
         │ │ │ │ │ │ •
         │ │ │ │ │ ├──
8 QED    │ │ │ │ │ │ B               6
         │ │ │ │ ├──
6 Cnj    │ │ │ │ │ A ∧ B             5
         │ │ │ ├──
5 CR     │ │ │ │ ⊥                   4
         │ │ ├──
4 RAA    │ │ │ ¬ B                   2
         │ ├──
2 Cnj    │ │ A ∧ ¬ B                 1
         ├──
1 PE     │ (A ∧ B) ∨ (A ∧ ¬ B)
```

```
         │ (A ∧ B) ∨ (A ∧ ¬ B)       1
         ├──
         │ │ A ∧ B                   2
         │ ├──
2 Ext    │ │ A                       (3)
2 Ext    │ │ B
         │ │ •
         │ ├──
3 QED    │ │ A                       1
         │ │ A ∧ ¬ B                 4
         │ ├──
4 Ext    │ │ A                       (5)
4 Ext    │ │ ¬ B
         │ │ •
         │ ├──
5 QED    │ │ A                       1
         ├──
1 PC     │ A
```

2. a.

```
         │ A ∨ A         1
         ├──
         │ │ A           (2)
         │ ├──
         │ │ •
         │ ├──
2 QED    │ │ A           1
         │ │ A           (3)
         │ ├──
         │ │ •
         │ ├──
3 QED    │ │ A           1
         ├──
1 PC     │ A
```

```
         │ A             (2)
         ├──
         │ │ ¬ A
         │ ├──
         │ │ •
         │ ├──
2 QED    │ │ A           1
         ├──
1 PE     │ A ∨ A
```

b.

```
         │ A ∨ B             1
         ├──
         │ │ A               (3)
         │ ├──
         │ │ │ ¬ B
         │ │ ├──
         │ │ │ •
         │ │ ├──
3 QED    │ │ │ A             2
         │ ├──
2 PE     │ │ B ∨ A           1
         │ │ B
         │ ├──
         │ │ │ ¬ A           (5)
         │ │ ├──
         │ │ │ •
         │ │ ├──
5 QED    │ │ │ B             4
         │ ├──
4 PE     │ │ B ∨ A           1
         ├──
1 PC     │ B ∨ A
```

```
         │ B ∨ A             2
         ├──
         │ │ ¬ A             (5)
         │ ├──
         │ │ │ B             (3)
         │ │ ├──
         │ │ │ •
         │ │ ├──
3 QED    │ │ │ B             2
         │ │ │ A             (5)
         │ │ ├──
         │ │ │ │ ¬ B
         │ │ │ ├──
         │ │ │ │ •
         │ │ │ ├──
5 Nc     │ │ │ │ ⊥           4
         │ │ ├──
4 IP     │ │ │ B             2
         │ ├──
2 PC     │ │ B               1
         ├──
1 PE     │ A ∨ B
```

c.

This is the second of the two derivations needed; the first appears in 4.2.3. In that one, disjunctive resources are exploited before disjunctive goals are planned for while the derivation at the left here illustrates the opposite approach.

	│ (A ∨ B) ∨ C	3
	│ │ ¬ A	(6)
	│ │ │ ¬ B	(8)
	│ │ │ │ A ∨ B	4
	│ │ │ │ │ A	(6)
	│ │ │ │ │ │ ¬ C	
	│ │ │ │ │ │ •	
6 Nc	│ │ │ │ │ │ ⊥	5
5 IP	│ │ │ │ │ C	4
	│ │ │ │ │ B	(8)
	│ │ │ │ │ │ ¬ C	
	│ │ │ │ │ │ •	
8 Nc	│ │ │ │ │ │ ⊥	7
7 IP	│ │ │ │ │ C	4
4 PC	│ │ │ │ C	3
	│ │ │ │ C	(9)
	│ │ │ │ •	
9 QED	│ │ │ │ C	3
3 PC	│ │ │ C	2
2 PE	│ │ B ∨ C	1
1 PE	│ A ∨ (B ∨ C)	

d.

	│ A ∨ (B ∧ ¬ B)	1
	│ │ A	(2)
	│ │ •	
2 QED	│ │ A	1
	│ │ B ∧ ¬ B	3
3 Ext	│ │ B	(5)
3 Ext	│ │ ¬ B	(5)
	│ │ │ ¬ A	
	│ │ │ •	
5 Nc	│ │ │ ⊥	4
4 IP	│ │ A	1
1 PC	│ A	

	│ A	(2)
	│ │ ¬ (B ∧ ¬ B)	
	│ │ •	
2 QED	│ │ A	1
1 PE	│ A ∨ (B ∧ ¬ B)	

e.

	│ ¬ (A ∨ B)	3,7
	│ │ │ A	(5)
	│ │ │ │ │ ¬ B	
	│ │ │ │ │ •	
5 QED	│ │ │ │ │ A	4
4 PE	│ │ │ │ A ∨ B	3
3 CR	│ │ │ ⊥	2
2 RAA	│ │ ¬ A	1
	│ │ │ B	(9)
	│ │ │ │ │ ¬ A	
	│ │ │ │ │ •	
9 QED	│ │ │ │ │ B	8
8 PE	│ │ │ │ A ∨ B	7
7 CR	│ │ │ ⊥	6
6 RAA	│ │ ¬ B	1
1 Cnj	│ ¬ A ∧ ¬ B	

	│ ¬ A ∧ ¬ B	1
1 Ext	│ ¬ A	(4)
1 Ext	│ ¬ B	(5)
	│ │ A ∨ B	3
	│ │ │ A	(4)
	│ │ │ •	
4 Nc	│ │ │ ⊥	3
	│ │ │ B	(5)
	│ │ │ •	
	│ │ │ ⊥	3
3 PC	│ │ ⊥	2
2 RAA	│ ¬ (A ∨ B)	

f.

	│ ¬ (A ∧ B)	3
	│ │ A	(5)
	│ │ │ B	(6)
	│ │ │ │ │ •	
5 QED	│ │ │ │ │ A	4
	│ │ │ │ │ •	
6 QED	│ │ │ │ │ B	4
4 Cnj	│ │ │ │ A ∧ B	3
3 CR	│ │ │ ⊥	2
2 RAA	│ │ ¬ B	1
1 PE	│ ¬ A ∨ ¬ B	

	│ ¬ A ∨ ¬ B	3
	│ │ A ∧ B	2
2 Ext	│ │ A	(4)
2 Ext	│ │ B	(5)
	│ │ │ ¬ A	(4)
	│ │ │ •	
4 Nc	│ │ │ ⊥	3
	│ │ │ ¬ B	(5)
	│ │ │ •	
5 Nc	│ │ │ ⊥	3
3 PC	│ │ ⊥	1
1 RAA	│ ¬ (A ∧ B)	

3. a.

```
        │ A ∨ B        2
        │ A
        ├──
        │ │ B
        │ ├──
        │ │ │ A
        │ │ ├──
        │ │ │ ○        A, B ⇏ ⊥
        │ │ ├──
        │ │ │ ⊥        2
        │ │
        │ │ │ B
        │ │ ├──
        │ │ │ ○        A, B ⇏ ⊥
        │ │ ├──
        │ │ │ ⊥        2
        │ ├──
2 PC    │ │ ⊥          1
        ├──
1 RAA   │ ¬ B
```

A B	A ∨ B , A / ¬ B
T T	Ⓣ Ⓣ Ⓕ

b.

```
         │ A ∨ (B ∧ C)        3,8
         ├──
         │ │ │ ¬ A            (5)
         │ │ ├──
         │ │ │ │ A            (5)
         │ │ │ ├──
         │ │ │ │ │ ¬ B
         │ │ │ │ ├──
         │ │ │ │ │ •
         │ │ │ │ ├──
5 Nc     │ │ │ │ │ ⊥          4
         │ │ │ ├──
4 IP     │ │ │ │ B            3
         │ │ │
         │ │ │ │ B ∧ C
         │ │ │ ├──
6 Ext    │ │ │ │ B            7
6 Ext    │ │ │ │ C
         │ │ │ │ •
         │ │ │ ├──
7 QED    │ │ │ │ B            3
         │ │ ├──
3 PC     │ │ │ B              2
         │ ├──
2 PE     │ │ A ∨ B            1
         │ │
         │ │ │ A
         │ │ ├──
         │ │ │ │ ¬ C
         │ │ │ ├──
         │ │ │ │ ○            A, ¬ C ⇏ ⊥
         │ │ │ ├──
         │ │ │ │ ⊥            9
9 IP     │ │ │ C              8
         │ │
         │ │ │ B ∧ C          10
         │ │ ├──
10 Ext   │ │ │ B              11
10 Ext   │ │ │ C
         │ │ │ •
         │ │ ├──
11 QED   │ │ │ C              8
         │ ├──
8 PC     │ │ C                1
         ├──
1 Cnj    │ (A ∨ B) ∧ C
```

Since entailment fails in one direction, equivalence must fail, so a second derivation for entailment in the other direction need not be pursued; but that entailment does hold, as is shown below.

```
         │ (A ∨ B) ∧ C        1
         ├──
1 Ext    │ A ∨ B              2
1 Ext    │ C                  (8)
         │
         │ │ A                (4)
         │ ├──
         │ │ │ ¬ (B ∧ C)
         │ │ ├──
         │ │ │ •
         │ │ ├──
4 QED    │ │ │ A              3
         │ ├──
3 PE     │ │ A ∨ (B ∧ C)      2
         │
         │ │ B                (7)
         │ ├──
         │ │ │ ¬ A
         │ │ ├──
         │ │ │ │ •
         │ │ │ ├──
7 QED    │ │ │ │ B            6
         │ │ │
         │ │ │ │ •
         │ │ │ ├──
8 QED    │ │ │ │ C            6
         │ │ ├──
6 Cnj    │ │ │ B ∧ C          5
         │ ├──
5 PE     │ │ A ∨ (B ∧ C)      2
         ├──
2 PC     │ A ∨ (B ∧ C)
```

Each of the following divides the one open gap:

A B C	A ∨ (B ∧ C) / (A ∨ B) ∧ C
T T F	Ⓣ F T Ⓕ
T F F	Ⓣ F T Ⓕ

c.

```
         │ ¬ (A ∨ B)          3
         ├──
         │ │ A                (5)
         │ ├──
         │ │ │ B
         │ │ ├──
         │ │ │ │ │ ¬ B
         │ │ │ │ ├──
         │ │ │ │ │ •
         │ │ │ │ ├──
5 QED    │ │ │ │ │ A          4
         │ │ │ │ ├──
4 PE     │ │ │ │ A ∨ B        3
         │ │ │ ├──
3 CR     │ │ │ ⊥              2
         │ │ ├──
2 RAA    │ │ ¬ B              1
         ├──
1 PE     │ ¬ A ∨ ¬ B
```

The following divide the first and second open gap, respectively:

A B	¬ A ∨ ¬ B / ¬ (A ∨ B)
F T	T Ⓣ F Ⓕ T
T F	F Ⓣ T Ⓕ T

```
         │ ¬ A ∨ ¬ B          2
         ├──
         │ │ A ∨ B            3,5
         │ ├──
         │ │ │ ¬ A            (4)
         │ │ ├──
         │ │ │ │ A            (4)
         │ │ │ ├──
         │ │ │ │ •
         │ │ │ ├──
4 Nc     │ │ │ │ ⊥            3
         │ │ │
         │ │ │ │ B
         │ │ │ ├──
         │ │ │ │ ○            ¬ A, B ⇏ ⊥
         │ │ │ ├──
         │ │ │ │ ⊥            3
         │ │ ├──
3 PC     │ │ │ ⊥              2
         │ │
         │ │ │ ¬ B            (6)
         │ │ ├──
         │ │ │ │ A
         │ │ │ ├──
         │ │ │ │ ○            A, ¬ B ⇏ ⊥
         │ │ │ ├──
         │ │ │ │ ⊥            5
         │ │ │
         │ │ │ │ B            (6)
         │ │ │ ├──
         │ │ │ │ •
         │ │ │ ├──
6 Nc     │ │ │ │ ⊥            5
         │ │ ├──
5 PC     │ │ │ ⊥              2
         │ ├──
2 PC     │ │ ⊥                1
         ├──
1 RAA    │ ¬ (A ∨ B)
```