4.2.4. The duality of conjunction and disjunction

While a conjunction and a disjunction formed from the same components are certainly not contradictories, the two connective are opposites in another sense, the one for which we have used the term *dual*.

This duality can be expressed in one way by saying that when conjunction and disjunction are applied to pairs of sentences whose corresponding components are contradictory, the results are contradictory. As an example, let us again take *X was home* and *X was out* to be contradictories. Now to get a sentence contradictory to *Ann and Bill were home*, we cannot take *Ann and Bill were out* since both sentences would be false if one of Ann and Bill was home and the other out. To get a contradictory to *Ann and Bill were home* we need to leave open those possibilities, and *Ann or Bill was out* will do this. That is, *Ann and Bill were home* is contradictory to *Ann or Bill was out* and, similarly, *Ann or Bill was home* is contradictory to *Ann and Bill were out*. And this is to say that ¬ *Ann and Bill were home* ⇔ *Ann or Bill was out* and that ¬ *Ann or Bill was home* ⇔ *Ann and Bill were out*.

When they are limited to the cases of contradictoriness captured by the bar notation, these patterns of equivalence are know as ***De Morgan's laws***:

$$\neg\,(\phi \wedge \psi) \Leftrightarrow \overline{\phi} \vee \overline{\psi}$$
$$\neg\,(\phi \vee \psi) \Leftrightarrow \overline{\phi} \wedge \overline{\psi}$$

Although these are named after Augustus De Morgan (1806-1871), they were known long before his time.

Another way to see the duality of conjunction and disjunction is to look at the principles that hold for them with respect to relative exhaustiveness. The table below follows the pattern of the one given for ⊤ and ⊥ in 1.4.6.

	as a premise	as an alternative
Conjunction	$\Gamma, \phi \wedge \psi \Rightarrow \Delta$ iff $\Gamma, \phi, \psi \Rightarrow \Delta$	$\Gamma \Rightarrow \phi \wedge \psi, \Delta$ iff both $\Gamma \Rightarrow \phi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$
Disjunction	$\Gamma, \phi \vee \psi \Rightarrow \Delta$ iff both $\Gamma, \phi \Rightarrow \Delta$ and $\Gamma, \psi \Rightarrow \Delta$	$\Gamma \Rightarrow \phi \vee \psi, \Delta$ iff $\Gamma \Rightarrow \phi, \psi, \Delta$

(Here *iff* is used as an abbreviation of *if and only if*.) Notice that the analogy between the upper left and lower right and between the lower left and upper right. That is, conjunction behaves as a premise in much the way disjunction behaves as an alternative and disjunction behaves as premise in much the way conjunction behaves as an alternative.

Since $\top$ and $\bot$ are paired as duals and so are conjunction and disjunction, you might wonder what serves as the dual to negation. In fact, it is dual to itself. If we negate each of a pair of contradictory sentences, the results are contradictory; that is, we do not need to apply different operations to the two contradictory sentences in order for the results to be contradictory. Notice also that the behavior of negation as a premise is analogous to its behavior as an alternative.

$$\Gamma, \neg \phi \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \phi, \Delta$$
$$\Gamma \Rightarrow \neg \phi, \Delta \text{ iff } \Gamma, \phi \Rightarrow \Delta$$

Having a negated premise or alternative is equivalent to having the unnegated sentence in the opposite role.

As was noted in 1.4.6, the term *duality* points to a certain sort of two-for-one principle. In particular, it is used when there is some way of associating vocabulary items as pairs so that replacing one member of a pair by the other throughout any truth will yield another truth. In our case, we have the associations

$\Rightarrow$ —— $\Leftarrow$

$\Leftrightarrow$ —— $\Leftrightarrow$

$\top$ —— $\bot$

$\neg$ —— $\neg$

$\wedge$ —— $\vee$

The equivalence arrow is dual to itself because it amounts to having both $\Rightarrow$ and $\Leftarrow$; and if each of them is reversed, we still have both.

Let us apply this association in a couple of examples. The principle

$$\Gamma, \phi \wedge \psi \Rightarrow \Delta \text{ iff } \Gamma, \phi, \psi \Rightarrow \Delta$$

(the principle on the upper left of the table above) turns into

$$\Gamma, \phi \vee \psi \Leftarrow \Delta \text{ iff } \Gamma, \phi, \psi \Leftarrow \Delta$$

which (once it is rewritten with premises on the left and alternatives on the right and the premises and alternatives re-ordered) amounts to

$$\Delta \Rightarrow \phi \vee \psi, \Gamma \text{ iff } \Delta \Rightarrow \phi, \psi, \Gamma$$

This last statement is the principle of the lower right of the table with the variables Γ and Δ interchanged. And the principle

$$\Gamma, \neg \phi \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \phi, \Delta$$

becomes

$$\Gamma, \neg \phi \Leftarrow \Delta \text{ iff } \Gamma \Leftarrow \phi, \Delta$$

or

$$\Delta \Rightarrow \neg \phi, \Gamma \text{ iff } \Delta, \phi \Rightarrow \Gamma$$

which is the second of the principles for negation stated above with Γ and Δ interchanged. In the next section, we will see more examples of such transformations between principles on the basis of dual concepts, but we have already seen other examples: each of the two forms of De Morgan's laws may be transformed into the other by this sort of association.