4.2.2. Proving disjunctions

Now let us look at disjunctions as conclusions. An entailment $\Gamma \Rightarrow \phi \vee \psi$ will hold if and only if $\phi \vee \psi$ is true in every possible world in which all members of Γ are true. But this is to say that at least one of ϕ and ψ is true in every such world, and that is a way of saying that Γ renders ϕ and ψ jointly exhaustive. So we can state the following principle:

$\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma \Rightarrow \phi, \psi$

Since the right-hand side has two alternatives, this is not a law concerning entailment alone, and we will not take the principle in this form as our account of the role of disjunctions as conclusions. However, we can use the basic law for relative exhaustiveness to restate the right-hand side as claim of entailment.

Indeed we have two ways of doing that. If ϕ and ϕ' are contradictory, we can say

$\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \phi' \Rightarrow \psi$

and if ψ and ψ' are contradictory, we can say

$\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \psi' \Rightarrow \phi$

In short, a disjunction is a valid conclusion from premises Γ if and only if adding to our premises a sentence contradictory to one disjunct enables us to validly conclude the other disjunct.

In stating a principle for disjunction we will limit ourselves to cases where a sentence and its negation are the pair of contradictory sentences. But, when the disjuncts are already negative, that leaves us with two choices for each of the pairs ϕ and ϕ' and ψ and ψ' since each of ϕ' and ψ' might be the result of either adding or dropping a negation. To avoid stating four principles to cover each of these possibilities, we will introduce some notation to capture the general idea of obtaining a contradictory sentence by either adding or dropping a negation. Let the notation $\bar{\phi}$ (read ϕ *bar*) stand for $\neg\, \phi$ when ϕ is not a negation and, when ϕ is the negation $\neg\, \chi$, for either $\neg\, \neg\, \chi$ or χ. That is, $\bar{\phi}$ is the result of either negating or, if possible, de-negating ϕ. We will say that $\bar{\phi}$ ***bars*** ϕ or is the ***barring*** of ϕ.

Then $\bar{\phi}$ and ϕ form a contradictory pair consisting of a sentence and its negation in one order or the other, so we may formulate our ***law for disjunction as a conclusion*** with only two statements:

(i) $\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \overline{\phi} \Rightarrow \psi$, and
(ii) $\Gamma \Rightarrow \phi \vee \psi$ if and only if $\Gamma, \overline{\psi} \Rightarrow \phi$

When these are implemented as derivation rules, they give us two ways of planning for a disjunctive goal. The two rules are shown as alternative developments in Figure 4.2.2-1. We will refer to both forms of the rule as ***Proof of Exhaustion*** (PE) since it is a way of showing that ϕ and ψ, taken together, exhaust all possibilities left open by the premises.

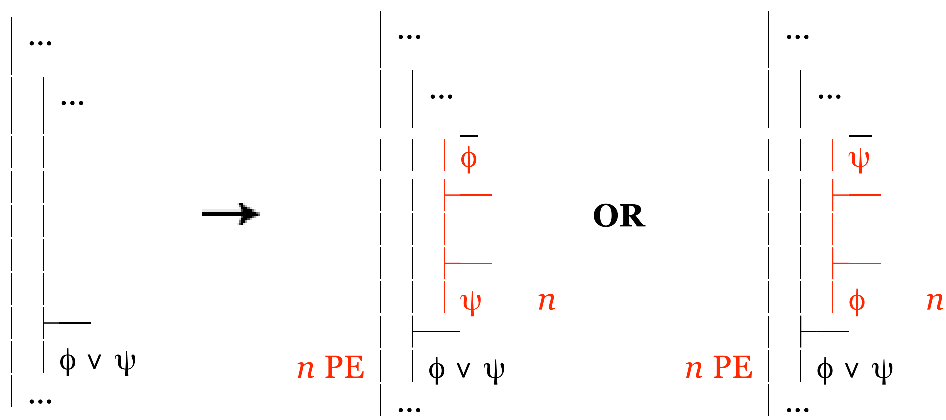

Fig. 4.2.2-1. Alternative ways of developing a derivation by planning for a disjunction at stage *n*.

In each way of developing a gap, we set one of the components of the disjunction as a new goal and add the barring (i.e., negation or de-negation) of the other component as a supposition. Both forms of planning will lead to the same answer in the end, but one or the other may be more efficient in a particular case. There is no simple way of predicting which choice is best but the following rules of thumb may help:

(i) if only one component is a negation, choose it to form the supposition (by dropping its negation);
(ii) if only one component is a non-negative compound choose it as the goal;
(iii) if only one component seems likely to figure in closing the gap and it is not a negation, choose it as the goal.

In many cases none of these suggestions will apply; but, in most such cases, neither one of the two forms of the rule is better than the other.

The supposition in PE may be described as ***hypothetical***, and this indicates a third role that suppositions may play. In *reductio* arguments and indirect proofs, we make suppositions with the aim of showing that they are false. In a proof by cases we make a pair of suppositions at least one of which we take to be true. In PE on the other hand, a supposition is made with no expectation of either truth or falsity. It is made instead simply to establish a connection between it and another claim. As we argue within the scope of the supposition, we are making a ***hypothetical argument***, one that explores the implications of the supposition in order to establish a connection between it and another claim. The conclusion we draw to end the scope of the supposition states this connection between the two claims. Here, it is $\phi \vee \psi$, so the connection between the two sentences is at least one of them is true. This is a statement made categorically; this, it no longer falls under the supposition.

There is some danger of getting tangled in the terminology here, so let's pause and look at it more closely. The terms *hypothetical* and *categorical* derive from an ancient classification of sentences into the "categorical," the "disjunctive," and the "hypothetical". Since disjunctions and "hypothetical sentences" (the conditionals to be studied in the next chapter) are ways of hedging claims, the term ***categorical*** has acquired the meaning 'unhedged'. Now the disjunctive goal to which we applied this term above certainly hedges each of its components, so it does not state them categorically. But, while sentences along the scope line of the hypothetical argument are stated only "under the hypothesis" that is the supposition of this argument, the disjunction following the argument is no longer hedged in this way, which means that it is stated categorically with respect to that supposition (though it may still fall in the scope of earlier ones). In short, when the scope line of a hypothetical argument ends, a hedged statement (of a possibly unhedged sentence) is converted into an unhedged statement of a sentence that incorporates a hedge.

As an example of this rule, consider the argument below, understanding *X was out* to be the denial of *X was home*. The validity of this argument can be established by the English derivation whose first stage is shown at the right.

Ann and Bill were not both home without the car being in the driveway
The car was not in the driveway
___
Either Ann or Bill was out

```
      │ ¬ ((A ∧ B) ∧ ¬ C)
      │ ¬ C
      ├──
      │ │ A
      │ ├──
      │ │
      │ ├──
      │ │ ¬ B                1
      ├──
1 PE  │ ¬ A ∨ ¬ B
```

The overall form is that of a hypothetical argument in which we suppose that Ann was at home (a supposition that is one of the two possibilities for ¬ A) and establish under this hypothesis that Bill was out. This shows the connection between Ann being out and Bill being out that we claim when we state categorically that at least one was out. When the hypothetical argument ends, we move from a statement of ¬ B under the hypothesis A to a statement of ¬ A ∨ ¬ B that is hedged by the added alternative ¬ A but that is no longer stated under the hypothesis A.

Notice that if we continue the derivation

```
        │ ¬ ((A ∧ B) ∧ ¬ C)
        │ ¬ C
        ├──
        │ │ A
        │ ├──
        │ │ │ B
        │ │ ├──
        │ │ │
        │ │ ├──
        │ │ │ ⊥                2
        │ ├──
2 RAA   │ │ ¬ B                1
        ├──
1 PE    │ ¬ A ∨ ¬ B
```

we plan for the goal ¬ B by supposing B for *reductio*. And this example illustrates the different functions of the two sorts of supposition. We suppose that Ann is home in order to show that ¬ B (*Bill is out*) is true in all possible worlds in which ¬ A (*Ann is out*) is false. We go on to show that ¬ B is true in these cases by showing that to suppose further that B would rule out all possibilities—i.e., that this supposition would be absurd when added to our premises and the supposition A. From one point of view, both suppositions are merely added assumptions. But we add the first in order to show that, by adding the second, we would go too far.