3.4.2. Some examples of consistency

The aim of this subsection is to consider a few examples, but its title makes a further general point. An interpretation that divides a dead-end open gap will divide a *reductio* argument and thus show that its premises can all be true together. That is, it will show that the active resources of a dead-end open gap form a consistent set. Counterexamples to arguments in chapter 2 did that, too, since they made all resources of the gap they divided true, but now that is the full significance of a counterexample since the goal of the gap it divides is ⊥ and is therefore automatically false.

Here is a simple example that exhibits a common pattern.

```
          │ ¬ (A ∧ B)          2
          │ A                  (4)
          ├──
          │ │ ¬ B
          │ ├──
          │ │ │ │ •
          │ │ │ ├──
4 QED     │ │ │ │ A            3
          │ │ │ │ │ ¬ B
          │ │ │ │ ├──
          │ │ │ │ │ ○          A, ¬ B ⇏ ⊥
          │ │ │ │ ├──
          │ │ │ │ │ ⊥          5
          │ │ │ ├──
5 IP      │ │ │ │ B            3
          │ │ ├──
3 Cnj     │ │ │ A ∧ B          2
          │ ├──
2 CR      │ │ ⊥                1
          ├──
1 RAA     │ B
```

A	B	¬ (A ∧ B)		, A	/ B
T	F	Ⓣ	F	Ⓣ	Ⓕ

It may seem odd to continue to stage 5 since, before IP is applied, the resources of the second gap are fully exploited and its goal is not among them. But remember, first, that gaps can now close by Nc, so we also need to be sure that there is no inconsistency among the resources. Of course, there is none here, and the use of IP could never introduce such an inconsistency unless the goal it is applied to was already among the resources. So, in this case, it really is clear before stage 5 that the gap will not close. But, with enough thought, it would have been clear before stage 1 that some gap would not close so the simple fact that a dead-end gap can be foreseen is not grounds for declaring one. A dead-end gap is an indication of failure made fully explicit. What we count as fully explicit is a conventional matter, and we will treat as fully explicit only what cannot be made more explicit by the system of derivations. In this case, that requires the final use of IP (though the closure of the first gap at stage 4 might have been ignored).

Here is a somewhat longer example. The derivation on the left represents the most straightforward approach, in which resources are exploited in the order in which they appear when there is a choice while the derivation at the right exploits the premises in the opposite order.

```
           │ ¬ (A ∧ ¬ B)                       3
           │ ¬ (A ∧ ¬ C)                       6, 11
           ├──
           │ │ B ∧ ¬ C                         2
           │ ├──
2 Ext      │ │ B
2 Ext      │ │ ¬ C                             (9),(14)
           │ │
           │ │ │ │ │ ¬ A
           │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ¬ A
           │ │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ○                 ¬ A, B, ¬ C ⇏ ⊥
           │ │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ⊥                 8
           │ │ │ │ │ │ ├──
8 IP       │ │ │ │ │ │ │ A                   7
           │ │ │ │ │ │
           │ │ │ │ │ │ │ •
           │ │ │ │ │ │ ├──
9 QED      │ │ │ │ │ │ │ ¬ C                 7
           │ │ │ │ │ ├──
7 Cnj      │ │ │ │ │ │ A ∧ ¬ C               6
           │ │ │ │ ├──
6 CR       │ │ │ │ │ ⊥                       5
           │ │ │ ├──
5 IP       │ │ │ │ A                         4
           │ │ │
           │ │ │ │ │ B
           │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ¬ A
           │ │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ○                 ¬ A, B, ¬ C ⇏ ⊥
           │ │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ⊥                 13
           │ │ │ │ │ │ ├──
13 IP      │ │ │ │ │ │ │ A                   12
           │ │ │ │ │ │
           │ │ │ │ │ │ │ •
           │ │ │ │ │ │ ├──
14 QED     │ │ │ │ │ │ │ ¬ C                 12
           │ │ │ │ │ ├──
12 Cnj     │ │ │ │ │ │ A ∧ ¬ C               11
           │ │ │ │ ├──
11 RC      │ │ │ │ │ ⊥                       10
           │ │ │ ├──
10 RAA     │ │ │ │ ¬ B                       4
           │ │ ├──
4 Cnj      │ │ │ A ∧ ¬ B                     3
           │ ├──
3 CR       │ │ ⊥                             1
           ├──
1 RAA      │ ¬ (B ∧ ¬ C)
```

```
           │ ¬ (A ∧ ¬ B)                       6
           │ ¬ (A ∧ ¬ C)                       3
           ├──
           │ │ B ∧ ¬ C                         2
           │ ├──
2 Ext      │ │ B
2 Ext      │ │ ¬ C                             (10)
           │ │
           │ │ │ │ │ ¬ A
           │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ¬ A
           │ │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ○                 ¬ A, B, ¬ C ⇏ ⊥
           │ │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ │ ⊥                 8
           │ │ │ │ │ │ ├──
8 IP       │ │ │ │ │ │ │ A                   7
           │ │ │ │ │ │
           │ │ │ │ │ │ │ B
           │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ ○                   ¬ A, B, ¬ C ⇏ ⊥
           │ │ │ │ │ │ ├──
           │ │ │ │ │ │ │ ⊥                   9
           │ │ │ │ │ │ ├──
9 RAA      │ │ │ │ │ │ │ ¬ B                 7
           │ │ │ │ │ ├──
7 Cnj      │ │ │ │ │ │ A ∧ ¬ B               6
           │ │ │ │ ├──
6 CR       │ │ │ │ │ ⊥                       5
           │ │ │ ├──
5 IP       │ │ │ │ A                         4
           │ │ │
           │ │ │ │ •
           │ │ │ ├──
10 QED     │ │ │ │ ¬ C                       4
           │ │ ├──
4 Cnj      │ │ │ A ∧ ¬ C                     3
           │ ├──
3 CR       │ │ ⊥                             1
           ├──
1 RAA      │ ¬ (B ∧ ¬ C)
```

A B C	¬ (A ∧ ¬ B) , ¬ (A ∧ ¬ C) / ¬ (B ∧ ¬ C)
F T F	Ⓣ F F Ⓣ F T Ⓕ T T

Although the full on the right derivation is significantly shorter than the one on the left, neither derivation needed to continue beyond stage 8, so the difference in this case was unimportant. However, in other cases, the order in which premises are exploited could matter more for length (though it will never matter for the final outcome). Notice also that the *reductio* we attempt to complete at stage 8 has the same supposition as the one we attempted at stage 5 and has fewer active resources (since another premise was exploited at a stage between the two). This is a consequence of the fact that the requirements for the second premise to be true are already met by other active resources when it is exploited at stage 6, so the exploitation ends up adding nothing to the resources. The repetition in the derivation is just the way this repetition in the content of the resources is made explicit.