3.1.4. Negated conjunctions and conjoined negations

While the ability to negate a negation does not enable us to say any more—however much more we can suggest—we increase the range of propositions we can express considerably when we mix negation and conjunction. The variety of English sentences whose forms we can express naturally will still be somewhat limited, and we will go on to capture others in the next two chapters. But the variety of logical relations between compounds and their components that can be expressed using conjunction and negation will be as great as any we will see when we are considering connectives alone (that is, until chapter 6). The real key to this power of expression lies in the ability to negate conjunctions, so let us look more closely at such forms.

We will begin with the example *It was not both hot and humid*.

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It was not both hot and humid
¬ *it was both hot and humid*
¬ (*it was hot* ∧ *it was humid*)

¬ (T ∧ M)
not both T and M

[T: *it was hot*; M: *it was humid*]

</div>

Parentheses record the fact that the sentence as a whole is a negation. That is, negation here has wider scope than conjunction and is thus the main connective.

We will refer to the way this sentence is related to its unanalyzed components as the ***not-both form***. Our analysis together with the truth tables for negation and conjunction enable us to calculate a truth table for it. The table below follows the conventions for exhibiting the values of compounds that were introduced in 2.1.7 . The plain roman Ts and Fs are the values for the conjunction ϕ ∧ ψ in each case, and the circled values for the form as a whole come by following the table for negation and taking the opposite of the value of the conjunction in each row. Each of the two columns of values on the right is written under the sign for the connective whose table was the last used in calculating it.

φ ψ	¬ (φ ∧ ψ)
T T	Ⓕ T
T F	Ⓣ F
F T	Ⓣ F
F F	Ⓣ F

In the symbolic analysis of the *not-both* form, parentheses not only reflect the structure of the sentence analyzed but also make a significant difference in the proposition expressed. If we drop them and write ¬ φ ∧ ψ (i.e., `both not` φ `and` ψ), we will no longer be marking the conjunction as a component of a larger negation. The negation sign will instead apply (by default) to φ alone, and the main connective will be conjunction. That is, we will have a conjunction whose first component is a negation. The truth table for this form is as follows:

φ ψ	¬φ ∧ ψ
T T	F Ⓕ
T F	F Ⓕ
F T	T Ⓣ
F F	T Ⓕ

In the example we began with, dropping the parentheses gives us ¬ T ∧ M (that is, `both not` T `and` M), which can be put into English as follows:

<div align="center">

¬ it was hot ∧ it was humid
It wasn't hot ∧ it was humid
It wasn't hot, but it was humid

</div>

And we will refer to the general form ¬ φ ∧ ψ as the **not-but *form***.

The *not-but* sentence above also could be expressed (though more awkwardly) as *It was both not hot and humid*. (If this does not seem to make sense, try reading *not hot* as if it was hyphenated and pause briefly after it; that is, read it as you would *It was both not-hot—and humid*.) A comparison of this last (awkward) expression of the *not-but* form with our original *not-both* example is revealing:

Sentence	*Analysis*
It was not both hot and humid	¬ (T ∧ M) ***or*** `not both` T `and` M
It was both not hot and humid	(¬ T ∧ M) ***or*** `both not` T `and` M

(The whole of the second analysis is parenthesized to make the comparison easier.) The order of the words expressing negation and conjunction in the two English sentences corresponds exactly to their order in the analysis written using English notation. In particular, the

word *both* can be seen to function in the English sentences, as it does in the analysis, to mark the beginning of the scope of a conjunction and thus to indicate whether the word *not* applies to the whole conjunction or only a part. Of course, things do not always work out this neatly in English, but the use of *both* after *not* is an important way of indicating exactly what is being denied. Emphasis is another way of indicating the scope of negation, and an emphasized *both*—as in *It was not both hot and humid*—can be particularly effective.

The real significance of negated conjunction lies in the way it modifies while combining, allowing us to say that at least one of the two components of the *not-both* form is false. The sentence *It was not both hot and humid* is false only when the components *It was hot* and *It was humid* are both true, so it leaves open every possibility in which at least one of them is false. And this is something we could not do by modifying the components separately and asserting each. On the other hand, a conjunction one or both of whose components is negative merely combines by adding content, and we could convey the same information by asserting the conjuncts separately.

While the *not-both* is the important new idea, conjunction of possibly negative components sometimes captures what we want to say; and there is a construction in English that seems designed to produce this logical form. The sentence *It was humid but it wasn't hot* could be rephrased as *It was humid but not hot* and thus as *It was humid without being hot*. So this last sentence, too, can be understood as a conjunction (i.e., as M ∧ ¬ T or `both` M `and not` T). Now *without* (in this use of the word) is a preposition, not a conjunction, so what follows it will not have the form of a sentence. But the object of *without* can be a nominalized predicate or nominalized sentence rather than an ordinary noun or noun phrase, and just about anything of the form φ ∧ ¬ ψ (which we will refer to as the **but-not form**) can be paraphrased using *without*. For example, *Sue listened but didn't respond* can be paraphrased as *Sue listened without responding*, and *Ann walked in but Bill didn't see her* could be paraphrased as *Ann walked in without Bill seeing her*. And, even when the object of *without* is an ordinary noun or noun phrase (rather than a nominalized predicate or sentence), the effect of *without* is often the same as that of a *but-not* form. Thus *Tom left without his coat* could be paraphrased as *Tom left but didn't take his coat* and thus analyzed as *Tom left* ∧ ¬ *Tom took his coat*. Of course, we have had to supply the verb *take* here, and we cannot expect any one pattern of paraphrase to work in all cases where *without* has an ordinary noun or noun phrase as its object.

Since this use of *without* is not a conjunction, it does not introduce a

second main verb; and this makes it especially convenient when we want to negate a *but-not* form. For the easiest way to express the negation of a whole sentence is to apply *not* to a single main verb. Suppose we wish to say something with the following form:

$$\neg \; (\textit{it will fall} \; \wedge \; \neg \; \textit{it will be pushed})$$
not both *it will fall* and not *it will be pushed*.

We might manage by expressing the three connectives one by one, ending with something like *It won't both fall and not be pushed*, where we have contrived a single conjoined predicate incorporating negation. But any such sentence is likely to be rather awkward. The natural way of making the claim analyzed above is to use *It won't fall without being pushed*. Accordingly, let us refer to the form ¬ (ϕ ∧ ¬ ψ) as the ***not-without* form**.

Of course, it is also possible to conjoin sentences both of which are negations. Indeed, *It was not hot and not humid* is sometimes an accurate description of the weather. We would analyze this symbolically as ¬ T ∧ ¬ M or both not T and not M. It will, at least for the time being to have a label for the form ¬ ϕ ∧ ¬ ψ, too; and the natural one is ***not-and-not* form**. Although this is an important sort of truth-functional compound, we will see another way of expressing it in the next chapter that is closer to the grammatical form usually taken by such compounds in English. For the more idiomatic way of say that is not hot and also not humid is with the sentence *It is neither hot nor humid*. We noted earlier that this sentence can be seen as a negation of *It is either hot or humid*, and its analysis along those lines will await our account of the word *or*. But, until we have that, the *not-and-not* form can serve as an analysis of *neither-nor* sentences since it has the right truth conditions.

This way of analyzing *neither-nor* sentences is not the only case where conjunction and negation can be used to analyze sentences we will be able to analyze more directly later. For example, many *if-then* sentences can be analyzed using the *not-without* form (though doing so may be jarring due to differences in implicatures). In fact, it is not hard to see that any truth-functional compound can be expressed using conjunction and negation alone.

Suppose the effect of some connective on the truth conditions of a sentence can be captured in a truth table—that is, suppose the connective is truth-functional. The force of a sentence formed by such a connective is to deny that the actual state (or history) of the world is described by any of the rows of the table in which the sentence is false. Now the description of the state of the world offered by a given row can

be captured by a run-on conjunction that affirms or denies each component in turn. For example, knowing that φ is assigned **T** and ψ is assigned **F** comes to the same thing as knowing that the sentence φ ∧ ¬ ψ is true. As a result, the compound sentence as a whole is equivalent to a conjunction of the denials of the sentences corresponding to each row in which the sentence is false. (At least this is so, if there are any such rows; otherwise, the sentence is a formal tautology and is equivalent to any other formal tautology, for example, ¬ (φ ∧ ¬ φ).)

To take a particular case, a compound with the table below can be thought of as saying that φ and ψ are not both truth and also that they are not both false, so it will be equivalent to ¬ (φ ∧ ψ) ∧ ¬ (¬ φ ∧ ¬ ψ).

φ ψ	
T T	F
T F	T
F T	T
F F	F

An English sentence whose grammatical form is close to this form—such as *Sam didn't eat both pie and cake, but he also didn't eat neither*—will be very cumbersome, and there are likely to be more idiomatic ways of saying the same thing whose most natural analyses would be different. But it is still important to note that it is possible to do say this sort of thing by putting the sentences *Sam ate pie* and *Sam ate cake* together using conjunction and negation alone. And that same idea can be applied no matter how many components the connective applies to and no matter what form the truth table takes. For this reason, conjunction and negation are said to form a ***truth-functionally complete*** set of connectives; that is, even with this limited vocabulary, we can express any truth function.