2.4.2. Lemmas for *reductio* arguments

We have seen that a lemma is bound to be safe if it is entailed by the goal we seek. That is, we can state following principle:

if $\phi \Rightarrow \psi$, then $\Gamma \Rightarrow \phi$ if and only if both $\Gamma \Rightarrow \psi$ and $\Gamma, \psi \Rightarrow \phi$

This tells us that when our desired conclusion ϕ implies a lemma ψ, we have not only the *if*-statement claim made by the law of lemmas but also the corresponding *only-if*-claim.

In principle, this idea can be applied to any case where the desired conclusion implies a sentence that might be useful in establishing it. We will build this into our system of derivations only in the case of the implications provided by EFQ, but implications provided by Ext are a better source of initial examples so we will first look at a use of that sort of lemma. Here is a derivation which uses the rule Lem to introduce a lemma that is the result of applying left Ext to the final goal.

```
         │ A ∧ B                              1
         ├──
1 Ext    │ A                                  (5),(9)
1 Ext    │ B                                  (4)
         │
         │ │ │ •
         │ │ ├──
4 QED    │ │ │ B                              3
         │ │ │ •
         │ │ ├──
5 QED    │ │ │ A                              3
         │ ├──
3 Cnj    │ │ B ∧ A                            2
         │ │ B ∧ A                            (7),(10)
         │ ├──
         │ │ │ •
         │ │ ├──
7 QED    │ │ │ B ∧ A                          6
         │ │ │ │ •
         │ │ │ ├──
9 QED    │ │ │ │ A                            8
         │ │ │ │ •
         │ │ │ ├──
10 QED   │ │ │ │ B ∧ A                        8
         │ │ ├──
8 Cnj    │ │ │ A ∧ (B ∧ A)                    6
         │ ├──
6 Cnj    │ │ (B ∧ A) ∧ (A ∧ (B ∧ A))          2
         ├──
2 Lem    │ (B ∧ A) ∧ (A ∧ (B ∧ A))
```

Here the rule Lem is applied at stage 2 with the left component of the goal as the lemma. This yields a slight shortening of the derivation since we are able to use the lemma to conclude B ∧ A by QED at stages 7 and 9 rather than repeating the proof used at stages 3-5 twice.

The simplification here is slight, and it occurs at all only because of a repetition in the goal that we would not expect to encounter often. While we would have more opportunities to use this sort of lemma in later chapters, there would not be enough to lead us to introduce a special rule, and this serves us only as an example. It is worth remembering, however, that it is legitimate pattern of deductive reasoning to first conclude one of the two components of a conjunction and then use that component to conclude the other (as we here have used the lemma B ∧ A to conclude A ∧ (B ∧ A)).

The pattern *Ex Falso Quodlibet* provides the basis for a much more important use of lemmas. An argument whose conclusion is ⊥ is often called a ***reductio argument***—where *reductio* is short for the Latin phrase *reductio ad absurdum* ('reduction to absurdity'). We will often need to use a lemma to complete such an argument and, since EFQ tells us that ⊥ entails any sentence, we know that any lemma we choose is safe. We will call the rule implementing this idea ***Lemma for Reductio*** or LFR:

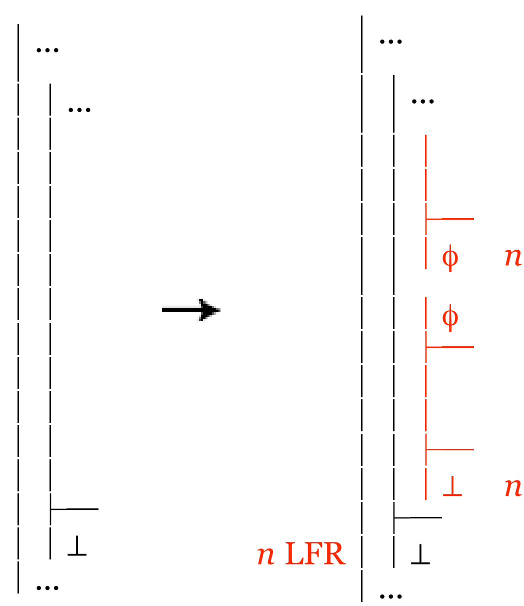

Fig. 2.4.2-1. Developing a derivation by introducing a lemma for a *reductio* at stage *n*.

We know this is safe from the general argument above, but it is also easy to see that directly. Any interpretation that divided either of the new gaps would certainly have to make all active resources of the original gap true; but an interpretation that did that would divide the original gap since the goal of that gap is ⊥ and is therefore bound to be false. So an interpretation cannot divide either of the new gaps unless it already divided their parent.

While this rule is certainly important, we are not yet in a position to illustrate it because, as yet, we have no non-trivial examples of formally valid *reductio* arguments. A *reductio* is formally valid only if its premises constitute a formally inconsistent set (that is, one whose members cannot be all true on any extensional interpretation), and the only formally inconsistent sets available with our current analyses of sentences contain ⊥ either as a premise or as a component of a premise. And we can show that such a set of premises entails ⊥ while using nothing but Ext and QED. This situation will change in the next chapter but, even there, our chief use of lemmas will be in a special modified version of LFR that is designed to exploit a resource at the same time it introduces a lemma.

That rule will be a direct rule, but rules that introduce lemmas usually will not be direct; and, in order to be sure a system employing them was decisive, we need to show that they can be considered progressive (on the right measure of distance from the end). The use of Lem illustrated first—i.e., a use where the lemma is a component of the goal—could be regarded as progressive provided we limited it to cases where the lemma was not already an active resource. But the second use of Lem—i.e., the use of LFR—can undermine decisiveness even if we forbid using a lemma that repeats a resource since the form of LFR places no constraints on the number of different lemmas that might be introduced. It would be enough to limit the lemmas that might be introduced to sentences that also appear as components of either active resources or goals, but weaker restrictions would also work. In general, we will not attempt to formulate the sort of restriction that would enable us to prove decisiveness for a system with LFR. The value of the rule is a practical one; and, in practice, the constraint of good sense in its use is restrictive enough.