2.4.1. The dangers of lemmas

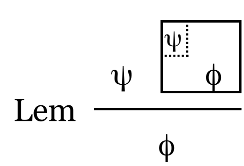
A fully general rule for introducing lemmas was cited in 2.3.2 as an example of an unsafe rule. That rule is unsafe because the lemma might not be a valid conclusion from our resources even though the conclusion we are trying the reach by way of it is a valid one. Such a rule would also prevent a system from being decisive because it would mean that a gap could always be developed further by introducing a lemma. However, as was noted earlier, more limited rules for introducing lemmas can be safe, and we will see that they can also be progressive. In this section we will look at the problems posed by lemmas more closely before considering a couple of special cases where these problems do not arise.

The law for lemmas of 1.4.2 can be stated as follows:

$\Gamma \Rightarrow \phi$ if both $\Gamma \Rightarrow \psi$ and $\Gamma, \psi \Rightarrow \phi$

Any possible world that is a counterexample to the first entailment will be a counterexample to one of the two on the right—the first of them if it makes ψ false and the second if it makes it true. So if both entailments on the right hold (that is, neither has a counterexample), the one on the left will hold, too. However, this principle holds only as an *if* claim; the corresponding *only if* does not hold in all cases. When $\Gamma \Rightarrow \phi$, we know that $\Gamma, \psi \Rightarrow \phi$ by the monotonicity of $\Rightarrow$; but, since ϕ and ψ need have no connection with one another, knowing that $\Gamma \Rightarrow \phi$ would by itself give us no reason to suppose that $\Gamma \Rightarrow \psi$. Of course, in a case where we know that that $\phi \Rightarrow \psi$, we know $\Gamma \Rightarrow \psi$ because of the chain law; and we may know $\Gamma \Rightarrow \psi$ in other cases because of special connections between Γ and ψ. These are the two sorts of situation in which we will use lemmas but, before turning to them, let's look at what a fully general (and unsafe) rule for lemmas would be like.

If used in tree-form proofs this rule might take the following form:



Here the proof of ϕ divides into two branches. The first is a proof of the lemma ψ and the second is a proof of ϕ using ψ in addition to the already available assumptions and the exploitation chains that grow from them. The box around the right-hand branch is intended to indicate that the use of ψ is limited to that part of the proof, and ψ appears at the upper left of that box to indicate that it is available for this branch as a further assumption from which we may begin exploitation chains.

In derivations, we will use scope lines to mark the scope of added assumptions. And such assumptions are marked off from other resources along the scope line by the sort of horizontal line we use to mark off the initial premises of a derivation.

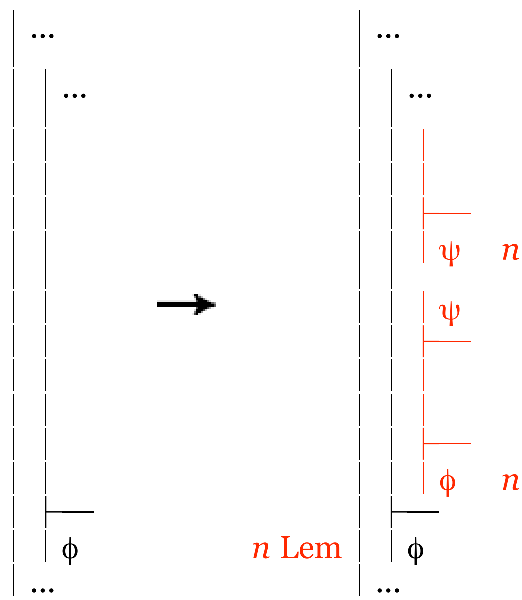


Fig. 2.4.1-1. Developing a derivation by introducing a lemma at stage *n* (a rule that will be part of our systems of derivations only in more restricted forms).

The assumption ψ is available only as long as the second of the two short scope lines continues, so it is effectively boxed off in derivations just as it is in the form of the rule displayed above for tree-form proofs.

The second of the two new gaps should be, if anything, easier to close than the original gap because it has a further resource. This increased ease is the point of introducing a lemma. The price we pay for this is the need to close the first new gap also. If the lemma is well chosen, reaching it may also be easier than closing the original gap by other means; but, because this rule is unsafe, we cannot be sure in general that the first new gap can be closed at all: ψ may not be a valid conclusion from its resources even if ϕ is. Because of this, the use of lemmas in ordinary deductive reasoning is accompanied by a readiness to backtrack, dropping the attempt to work by way of the lemma and looking for another approach to the proof. The notation of derivations is not designed to incorporate backtracking, so we will use lemmas only in cases where we can be sure there will be no need to do that.

Even in cases where we can sure backtracking is not necessary, the introduction of lemmas can interfere with decisiveness because it is in principle possible to keep introducing safe lemmas forever. In ordinary deductive reasoning, that problem is solved by good sense (or, if that's absent, by mortality). We will rely on those, too; but the system of derivations is designed to make the constraints on deductive reasoning explicit, so we will also consider the ways in which the use of safe lemmas might be limited further.