1.4.5. Exhaustiveness

Exclusion has entailment as its natural opposite but inconsistency has a natural opposite of another sort. We will say that a set is ***exhaustive*** or that its members are ***jointly exhaustive*** under the following conditions:

Γ is exhaustive	if and only if	there is no possible world in which all members of Γ are false
	if and only if	in each possible world, at least one member of Γ is true

The term *exhaustive* reflects the fact that an exhaustive set exhausts all possibilities in the sense that any possible world is left open by at least one member of the set. That is, if we collect the possible worlds left open by each of the members of an exhaustive set and combine all these collections, we will find all possible worlds included.

Exhaustiveness is an unconditional guarantee but it applies to sentences, not individually, but as a group. That is the reason for using the qualification *jointly* when we speak about the members of the set rather than the set itself. And we often have reason to speak of the members because the most important application of this idea for our purposes is to sets with two members. In that case, we say pair of sentences ϕ and ψ are jointly exhaustive when the set formed of the two is exhaustive—that is, when we are guaranteed that at least one of the two is true.

The most important application of the joint exhaustiveness of pairs is in the analysis of the relation of ***contradictoriness***, which provides a kind of opposite to equivalence:

ϕ and ψ are contradictory	if and only if	there is no possible world in which ϕ and ψ have the same truth value
	if and only if	in each possible world, ϕ and ψ have opposite truth values

To say that a pair of sentences come with this sort of guarantee of opposite truth values is to say that we have a guarantee that they are not both true and a guarantee that they are not both false. That is, for any sentences ϕ and ψ,

ϕ and ψ are contradictory if and only if ϕ and ψ are both mutually exclusive and jointly exhaustive (***basic law for contradictoriness***)

Although in ordinary discourse, the term *contradictory* is often applied to sentences that are merely mutually exclusive, in logical contexts it tends to be applied only to sentences that are also jointly exhaustive. Contradictoriness will play a central role in our account of the logical properties of negation and it is crucial for this that it have ties to both inconsistency and exhaustiveness.

The final deductive concept we will consider is a very general relation that is both a conditional guarantee related to exhaustiveness and a generalization of both entailment and inconsistency. ***Relative exhaustiveness*** is a relation between sets of sentences; when it holds, we will say that one set ***renders*** the other set ***exhaustive***. Our notation for this idea will extend the use of the entailment arrow to allow a set or a list of sentences to appear on the right. Relative exhaustiveness is defined as follows:

Γ ⇒ Δ	if and only if	there is no possible world in which all members of Δ are false while all members of Γ are true
	if and only if	in each possible world in which all members of Γ are true, at least one member of Δ is true

When a set Δ is exhaustive relative to a set Γ (that is, when Γ ⇒ Δ) the collection containing of any possible world left open by any member of Δ includes all worlds in which every member of Γ is true. Notice that this is quite different from saying that Γ entails each member of Δ (a relation between sets mentioned in 1.4.2) for that would imply a conditional guarantee that all members of Δ are true while relative exhaustiveness provides instead a guarantee that at least one member of Δ is true. For this reason, we will refer to multiple sentences on the right of ⇒ as ***alternatives*** rather than conclusions. In these terms, the definition of relative exhaustiveness tells us that a set of premises renders a set of alternatives exhaustive if and only if, in each possible world in which all the premises are true, at least one of the alternatives is true. Let us extend the idea of division from 1.4.1 to pairs of sets, saying that a possible world ***divides*** Γ from Δ when each member of Γ is true in that world while each member of Δ is false. Then we can say that Γ ⇒ Δ when there is no possible world that divides Γ from Δ. So a world that divides Γ from Δ is a counterexample to exhaustiveness of Δ relative to Γ.

There are three basic principles for relative exhaustiveness, which are rough analogues of the laws for implication and entailment. For any sentence ϕ and any sets Γ, Δ, Σ, and Θ of sentences:

$\phi \Rightarrow \phi$
if $\Gamma \Rightarrow \Delta$, then $\Gamma, \Sigma \Rightarrow \Delta, \Theta$
if $\Gamma \Rightarrow \phi, \Delta$ and $\Gamma, \phi \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$

First corresponds to the reflexivity of implication and the law for premises, and the second corresponds to the law of monotonicity. The third—usually called the ***cut law***—is related to both the chain law and the law for lemmas (which are closely related to each other).

The second of these principles reflects the fact that being able to divide sets means being able to assign certain values to all their members. As a result, if it is impossible to do this for given sets Γ and Δ, it will remain impossible if members are added to either of them. The first principle reflects the fact that, because a sentence cannot be both true and false, it cannot be divided from itself. The cut law reflects the other side of the coin, the fact every sentence is either true or false. The easiest way to see the connection involves a kind of roundabout argument that is one of the reasons that the negative forms of definitions are useful. Suppose that $\Gamma \Rightarrow \Delta$ fails—i.e., that some possible world divides Γ from Δ. This world must also assign some truth value to any given sentence ϕ. If it makes ϕ false while dividing Γ from Δ, the claim $\Gamma \Rightarrow \phi, \Delta$ will fail; and, if it makes ϕ true, the claim $\Gamma, \phi \Rightarrow \Delta$ will fail. Thus, if $\Gamma \Rightarrow \Delta$ fails, then so will either $\Gamma \Rightarrow \phi, \Delta$ or $\Gamma, \phi \Rightarrow \Delta$. But that means that if $\Gamma \Rightarrow \phi, \Delta$ are $\Gamma, \phi \Rightarrow \Delta$ hold, the claim $\Gamma \Rightarrow \Delta$ must hold, too.