1.2.2. Truth conditions and propositions

If we use the terminology of the last section, we can say that the conclusion of an inference brings with it no new risk of error when there is no possible world in which it has the truth value **F** while the premises all have the value **T**. This means that, when judging whether a relation of entailment holds between premises Γ and a conclusion ϕ, it is enough know the truth value of ϕ and the truth values of the members of Γ in each possible world. We will assume that a sentence always has one or the other of the truth values **T** and **F**, so we will know the truth value of a sentence in each possible world if we know which possible worlds are the ones in which it is true. We will refer to this aspect of the meaning of a sentence as the ***proposition*** it expresses.

A proposition is thus an abstract entity that encapsulates the aspects of a sentence's meaning which are relevant to questions of entailment (and deductive reasoning generally). The relations of entailment that hold between sentences derive from relations among the propositions they express. This can help in thinking about the analogies between mathematics and logic: deductive logic is concerned with propositions in the way algebra and the theory of functions are concerned with numbers. Since the relation of entailment depends only on the propositions expressed by sentences, we will use the term *entailment* both for a relation between sentences and for the corresponding relation between the propositions expressed by them. And this relation among propositions provides the fundamental order for the range of propositions in the way that the relation ≤ provides the fundamental order of the numbers. We will look further at this analogy between propositions and numbers in 1.2.3, but first we will look in more detail at the idea of specifying a truth value in each of the various logically possible worlds.

The proposition expressed by a sentence determines its truth value in each of the full range of logical possibilities and, in this sense, specifies its ***truth conditions***. A sentence leaves open a certain range of possibilities, those in which it is true, and rules out others, those in which it is false. Comparing the ranges of possibilities left open and ruled out by sentences—i.e., comparing their truth conditions—provides a way of comparing the their contents, the strengths of the claims they make. Consider the successively more specific statements:

The package will arrive sometime
The package will arrive next week
The package will arrive next Wednesday

All leave open many possibilities since they say nothing about, for example, the time of day the package will arrive, the form of transportation used, or its condition on arrival (to say nothing of the myriad of other bits of information in a full possible history of the universe); but each statement rules out some possibilities left open by the statement above it. And in general, the less information contained by some statement, the more possibilities it leaves open; the more it says, the more possibilities it rules out.

Let us look at entailment from this perspective. Begin with our recent characterization of risk-free inference: a sentence ϕ is entailed by a set Γ if there is no possible world in which ϕ is false while all members of Γ are true. This is to say that ϕ does not rule out any possibility that is left open by all members of Γ. Now, a possibility is left open by all members of Γ if there is no member that rules it out, so we can say that ϕ is entailed by Γ if it does not rule out any possibility that is not already ruled out by at least one member of Γ. If we think of possibilities ruled out as an indication of content, this amounts to the claim that ϕ is entailed by Γ if its content does not extend beyond the content already found among the members of Γ. And that brings us back to another characterization of deductive inference—namely, as the extraction of information, for we are saying that ϕ is entailed by Γ if its informational content is already to be found among the members of Γ taken together.

Of course, two sentences may provide exactly the same information, differing only in their wording. For example, although one of the sentences *Sam lives somewhere in northern Illinois or southern Wisconsin* and *Sam lives somewhere in southern Wisconsin or northern Illinois* might be chosen over the other depending on the circumstances, they rule out exactly the same possibilities for Sam's residence and thus express the same proposition. We will say that sentences that have the same informational content are ***(logically) equivalent*** (usually dropping the qualification *logically* since we will not be considering other sorts of equivalence).

Logically equivalent sentences divide logical space between possibilities ruled out and possibilities left open in exactly the same way. Since they have the same content they are valid conclusions from the same premises and can be freely substituted for each other as premises of an argument without changing its validity. In particular, logically equivalence sentences will entail each other. Also, when sentences are not logically equivalent, there must be some possible world where one is true and the other false. This cannot happen if a pair of sentences entail each other, so any pair of sentences that entail each other must be logically equivalent. Our notation for logical equivalence—the sign $\Leftrightarrow$

(***left right double arrow***)—reflects this characterization of equivalence as mutual entailment.

There are two extreme examples of truth conditions or propositions. A sentence that is true in all possible worlds says nothing. It has no informational content because it leaves open all possibilities and rules nothing out. For example, the weather "forecast" *Either it will rain or it won't* has no chance of being wrong and is, therefore, completely worthless as a prediction. We will say that such a sentence is a ***tautology***. Although there are many (indeed, infinitely many) tautologies, all express the same proposition; and the words that they use to express it are beside the point since they all say nothing in the end. That is, any two tautologies are logically equivalent. It is convenient to distinguish a particular tautology and mark it by special notation, which is specified simply as a sentence that says nothing. We will call this sentence ***Tautology*** and use the sign ⊤ (***down tack***) as our notation for it. Notice that ⊤ entails nothing except other tautologies (since any other sentence would add information) and is entailed by any set of premises (because it will not add information to any set of sentences).

At the other extreme of truth conditions is a sentence that rules out all possibilities. The fact that this is the opposite of tautology might suggest that it is maximally informative, but it sets an upper bound on informativeness in a different way: any genuinely informative sentence must say less. The ultimate aim of providing information is to narrow down possibilities until a single one remains, for this would provide a complete description of the history of the universe. To go beyond this would leave us with nothing because there is no way to distinguish among possibilities if all are ruled out. For example, the forecast *It will rain, but it won't* is far from non-committal since it stands no chance of being right, but it is no more helpful than a tautologous one. Sentences like this make logically impossible claims and we will refer to them as ***absurd***. As was the case with tautologies, any two absurd sentences are logically equivalent. Also as with tautologies, we will introduce a particular example of an absurdity, named ***Absurdity***, and we use the special notation ⊥ (the perpendicular sign, or ***up tack***) for it. Taken as a single premise, ⊥ entails every sentence. And that is another way of indicating the problem with absurd sentences: they try to go off in every direction at once. A set of premises that entails ⊥ need not contain an absurd sentence but, taken together, the sentences of such a set rule out all possibilities, and the set is therefore equally problematic. Although such a set is itself useless as a source of information, the idea of set of sentences whose members cannot all be true together will prove to be important, and ⊥ will serve us by helping us to characterize this sort of set in terms of entailment.

If we add Tautology and Absurdity to our earlier English examples, we get the following series of successively stronger claims:

$\top$
The package will arrive sometime
The package will arrive next week
The package will arrive next Wednesday
$\bot$

It's not hard to find propositions that lie between those expressed by each successive pair here. In particular, a proposition that leaves open only one of the possible worlds in which the package arrives next Wednesday would lie between *The package will arrive next Wednesday* and $\bot$. As before, each sentence entails itself and any above it. It is worth noting that, in this example, apart from the relations involving $\top$ and $\bot$, the relations of entailment will not follow from the laws we will consider in later chapters. For those laws will concern the logical words *and*, *not*, etc., while these entailments depend on connections among the time designations *next Wednesday*, *next week*, and *sometime*.