1. Problem 11

This problem was suggested by John Maharry.

Here are a couple questions about decimal representations of fractions. If p/q is a fraction (in lowest terms) then the decimal representation of the fraction p/q must eventually terminate or start repeating. Can you determine what properties about p and q make a fraction that terminates? If the decimal representation repeats, the repeating sequence might not start right at the beginning (e.g. 679/5500=0.123454545...). Can you determine properties about p and q that determine how many decimal places occur before the repeating pattern begins and how many decimal places are in the repeating pattern? Can you find fractions (in lowest terms) that equal 0.145314531453..., 0.9871111111... and 0.200812121212...?

2. Solution

(For an explanation of why rational numbers have repeating decimal expansions, see the next section.)

A fraction p/q (in lowest terms) will terminate iff $q = 2^m 5^n$ for some m and n. For such q, we can multiply the top and the bottom by an appropriate common factor (such as $2^n 5^m$) to get a power of 10 in the denominator. For any other q, it is impossible to rewrite the fraction with a power of 10 in the denominator, so p/q has no terminating decimal representation. (In any base, it is the prime factors of the base that are relevant. So, in base 60, the "decimal" representation of p/q terminates iff q has only 2, 3, and 5 as prime factors.)

For the remaining questions, we first make some observations. Fractions of the form $p/9$, $p/99$, $p/999$, etc. (not necessarily in lowest terms), give decimals that repeat every n digits, where n is the number of 9s. E.g., $12/99 = 0.121212121212\ldots$; $321/999 = 0.321321321\ldots$. (To see why this is the case, consider a repeating decimal like $0.121212\ldots$. This is $0.12 + 0.12/100 + 0.12/100^2 + 0.12/100^3 + \cdots$, a geometric series with first term 0.12 and ratio 1/100, so it sums to $0.12/(1-1/100) = 12/(100-1) = 12/99$.) Now, if we take such a fraction, say 12/99, and divide by a power of 10, say 1000, then we get a few zeros before the decimal repeats: $12/99000 = (12/99)/1000 = 0.000121212\ldots$. If we then add in some $p/1000$, we can fill in those zeros with something else; e.g., $0.567121212\ldots = 567/1000 + 12/99000 = (567 \cdot 99 + 12)/99000$. So, any repeating (but nonterminating) decimal can be written in the form p/q where q, when written in base 10, is a sequence of one or more 9s followed by zero or more 0s (such as 99000, 9, 900, 9999999000000000000, etc.).[1] Let $9_i 0_j$ denote the number consisting

[1] An interesting consequence of this is that every positive integer m can be multiplied by some n such that mn takes the form of one or more 9s followed by zero or more 0s. For instance, 35 can be multiplied by 285714 to get 9999990. This carries over to other bases by the same reasoning; e.g., in base 6, every m can be multiplied by some n such that mn has some number of 5s followed by some number of 0s.

of i 9s followed by j 0s; e.g., $9_5 0_3 = 99999000$. Say that p/q is in *pseudo-lowest terms* if we have q in this form, with as few 9s and 0s as possible. E.g., 1110/9990 is not in pseudo-lowest terms because we can write it as 1/9, but 12345/99000 is in pseudo-lowest terms because, although it can be reduced to 823/6600, 12345/99000 is the simplest representation where the denominator takes the form of $9_i 0_j$.

We can now answer the main question. Given any fraction p/q, rewrite p/q in pseudo-lowest terms to get p'/q', where $q' = 9_i 0_j$. Then, by reasoning much like the previous paragraph, the decimal expansion of p/q will have a repeating sequence of i digits beginning j digits after the decimal point. For example, the number 12/37 is, in pseudo-lowest terms, 65684210526315789408/99999999999999999900 (the denominator is $9_{18} 0_2$), so its decimal expansion has a repeating pattern of 18 digits beginning two digits after the decimal point: $0.65\overline{684210526315789473}$. The value of j is fairly easy to determine: if the prime factorization of q is $2^m \cdot 5^n \cdot$ (other primes), then j is the larger of m and n. There appears to be no simple way to determine i short of calculating the decimal expansion of p/q, but perhaps I am mistaken.

As for the specific examples offered:

$$\begin{aligned}
0.145314531453\ldots &= 1453/9999 \\
0.9871111111\ldots &= 987/1000 + 1/9000 = (987 \cdot 9 + 1)/9000 \\
&= 8884/9000 = 2221/2250 \\
0.200812121212\ldots &= (2008 \cdot 99 + 12)/990000 \\
&= 198804/990000 = 16567/82500
\end{aligned}$$

3. Why rational numbers have repeating decimal expansions

Suppose you are computing the decimal expansion of p/q by long division. At each step, you have a remainder that is a nonnegative integer less than q, and the further steps of the division are determined by this remainder. Since there are only finitely many numbers between 0 and q, we must eventually get the same remainder r twice; from that point onward, the sequence of remainders will be the same as it was the first time we had r as a remainder. So the expansion repeats. (If we ever hit remainder 0, then the decimal terminates, or repeats 0000..., depending on your point of view.) For example, when calculating 3/7, the sequence of remainders is $3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, \ldots$; when the remainders repeat, the digits in the decimal expansion repeat: $0.428571428571\ldots$.