1. Problem 10

You wish to break into a room whose entry is restricted by a keypad. (This is all strictly on the up and up: an evil but not particularly ambitious genius has stolen your wallet, and put it in this room.) The keypad has six buttons (numbered 1–6), which light up after being pressed; if the correct set of buttons is lit, the door will unlock with an audible click and you can retrieve your wallet.There is also a reset button, which returns all buttons to the unlit state. What is the shortest sequence of keypresses you can find, including presses of the reset button, that guarantees you unlock the door?

2. Solution

This solution was submitted by Randy Berta '76, and appears here with his permission, with minor editing.

The formula for calculating the number of combinations of size r from a set of n elements is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

We have combinations of 6 elements of all possible lengths, 0 through 6, so the total number of possible combinations to open the door is

$$\binom{6}{0}+\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6} = 1+6+15+20+15+6+1 = 64.$$

In our first pass we try 123456 (and assuming that fails, reset). We eliminate our 6 digit combination, and one of each of our shorter combinations. 1 combination of 7 keystrokes = 7 keystrokes

Next we attack the 5 remaining 5 digit combinations using sequences that do not repeat (in the initial keystrokes) any of the shorter sequences already eliminated. This will eliminate all of the 5-digit and all of the 1-digit combinations, leaving 9 2-digit, 14 3-digit, and 9 4-digit combinations. 5 combinations of 6 keystrokes (5 plus reset) = 30 keystrokes

Next we attack the 9 remaining 4-digit combinations, again entering them in an order so as to avoid entering any of the previously eliminated 3-digit or 2-digit sequences. This will eliminate all but 5 3-digit sequences. 9 combinations of 5 keystrokes (4 plus reset) = 45 keystrokes

Finally we enter the 5 remaining 3-digit combinations, and, assuming we get it right on the fifth and final sequence, we will press reset 4 of the 5 times. 5 combinations of 4 keystrokes (minus one) = 19 keystrokes

In total we would have to press 7 + 30 + 45 + 19 = 101 keystrokes to guarantee opening the door. A particular example of such a sequence: 123456R61234R56123R45612R34561R23456R3614R3561R3526R2634R2461 R2456R1425R1352R1345R461R346R256R251R154.