1 Problem

Find a set A of nonnegative whole numbers, with as few elements as possible, so that all the numbers 0,1,2,3,...,100 can be written as $c + d$, for some c and d in A. (You are allowed to have $c = d$, so if 5 is in A, you can get $10 = 5 + 5$.) The winning entry will be the smallest such set.

(This problem is due to John Connett.)

2 Solution

There are many ways to get a set with only 19 elements. One such set would be $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$. This gives us every number $0, \ldots, 100$ because 100=10+90, and a two-digit number such as 75 can be written as 75=70 + 5.

Observe, though, that if we had included 15 in the set instead of 5, we could have written $75 = 60 + 15$. With this idea, for each digit n, we can try including *either* n or $1n$ in our set (e.g., we might have 6 or 16 in our set). If we can find a way of picking these 10 numbers such that they cover enough of of what is below 30, then we can let the remaining 8 numbers be 20, 30, 40, 50, 60, 70, 80, 90, and get a set with 18 elements that covers everything up to 100. With a little trial and error, we can come up with 0,1,3,4,6,8,12,15,17,19; of the numbers below 30, these miss 26 and 28, but we get these after 20 is included. So, let $A = \{0, 1, 3, 4, 6, 8, 12, 15, 17, 19, 20, 30, 40, 50, 60, 70, 80, 90\}$, a set of 18 elements. As seen above, these cover everything below 30. Numbers mn (we are forming the two-digit number $10m + n$ here, not the product mn) above 30 are covered as follows: either n or $1n$ must be in our set; if n is in our set, we have $mn = m0 + n$, and if $1n$ is in our set, we have $mn = m'0 + 1n$, where $m' = m - 1$. For example, 46 can be written as $40 + 6$, while 47 can be written as $30 + 17$.

There are are even a few ways to get by with only 17 elements. Here are all of them:

$$\{0, 1, 3, 4, 7, 8, 9, 16, 17, 21, 24, 35, 46, 57, 68, 79, 90\}$$
$$\{0, 1, 3, 4, 5, 8, 14, 20, 26, 32, 38, 44, 45, 47, 48, 49, 52\}$$
$$\{0, 1, 3, 4, 5, 8, 14, 20, 26, 32, 38, 44, 46, 47, 48, 49, 51\}$$
$$\{0, 1, 3, 4, 5, 8, 14, 20, 26, 32, 38, 44, 46, 48, 49, 51, 53\}$$
$$\{0, 1, 3, 4, 5, 8, 14, 20, 26, 32, 38, 44, 47, 48, 49, 51, 52\}$$

For comparison, there are 19508 solutions with 18 elements. There are no solutions with 16 or fewer elements.

The first of the 17-element solutions above could conceivably be found by a human: it is like our 18-element solution, except that it works in base 11 (we have 0,1,3,4,7,8,9,16,17,21 up front, and then several numbers spaced 11 apart: 24,35,46,57,68,79,90). It also requires one more trick. You might notice that none of the first 10 elements is equal to 2 modulo 11, suggesting that we will have trouble getting numbers like 48, 59, 70, 81, and 92. However, these numbers can be obtained as sums within the last 7 elements. For example, 59=24+35; 92=46+46=35+57=24+68.

The solutions with fewer than 19 elements were all found with the help of a computer (the difficulty lies in searching efficiently; simply going through all possible 17-element sets would take 210 years, assuming optimistically that we could go through 1 billion sets per second).