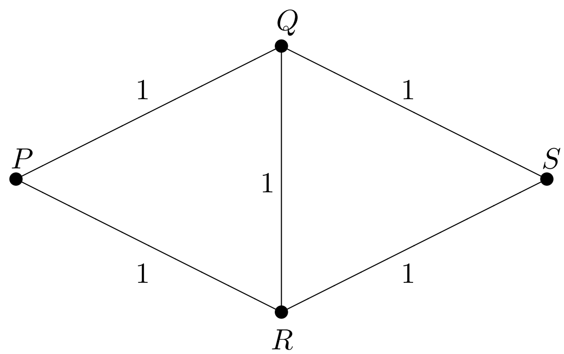
1 Is three colors enough?

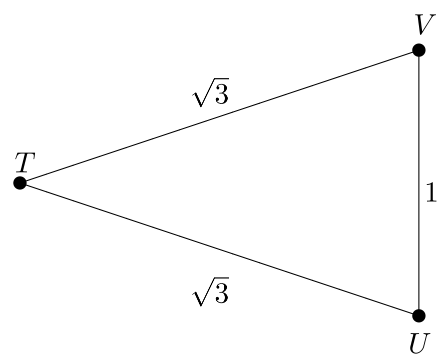
Suppose that you color every point in the plane using one of three colors. Is it possible that no two points a distance of 1 unit apart have the same color?

No, it is not possible. Suppose that we were able to color the plane with 3 colors such that all pairs of points 1 unit apart were colored differently. Consider the following diagram.



Q and R must have two different colors, leaving only one possible color for P and S—that is, P and S must have the same color. The distance from P to S is $\sqrt{3}$, and we could draw this diagram anywhere in the plane, so any pair of points $\sqrt{3}$ units apart must have the same color.

Now consider the following triangle.



Because T and U are $\sqrt{3}$ units apart, they must have the same color; similarly, T and V must have the color. So U and V must have the same color. This contradicts our assumption that all pairs a unit apart are colored differently. So, it cannot be done with 3 colors.

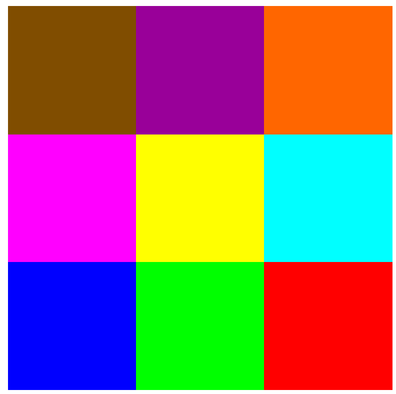


Figure 1: Coloring for a square tile

2 How many colors are needed?

2.1 Nine is enough

One can color the plane with 9 colors so that points 1 unit apart are colored differently, as follows. Tile the plane with squares with sides of length 2. Each of these squares will be broken into 9 smaller sqaures and colored as in Figure 1 (for points that are on the edge between two of the smaller squares, use either color).

Each of the smaller squares has sides of length 2/3. Consider the dark blue square. Its diagonal has length $2\sqrt{2}/3 \approx 0.94$, which is less than 1, so no two points within that square have distance exactly one from each other. On the other hand, every point within this square has a distance of at least 4/3 from any point in any other dark blue square. So, no two points 1 unit apart are both dark blue. The same can be said for any of the other colors. So, no two points 1 unit apart get the same color.

2.2 Seven is enough

Using essentially the same idea as above, but with hexagons instead of squares, one can get away with 7 colors, as shown in Figure 2.

2.3 Fewer than 7?

It is unknown whether or not the plane can be colored with fewer than 7 colors (so that no points 1 unit apart get the same color). The best that is known is that it requires at least 4, and at most 7. The problem of

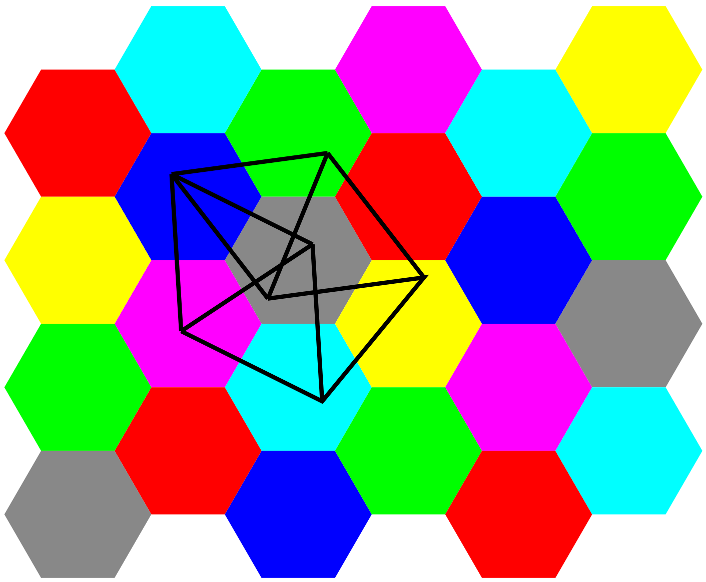


Figure 2: Tiling with hexagons. (This image has been released into the public domain by its author, David Eppstein.)

determining exactly how many are needed is known as the *Hadwiger-Nelson problem*.