PLANIMETERS AND ISOPERIMETRIC INEQUALITIES ON CONSTANT CURVATURE SURFACES

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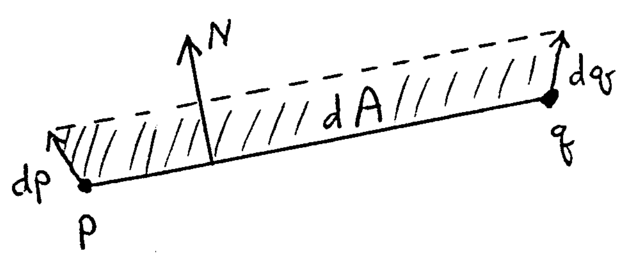
Preliminary Version

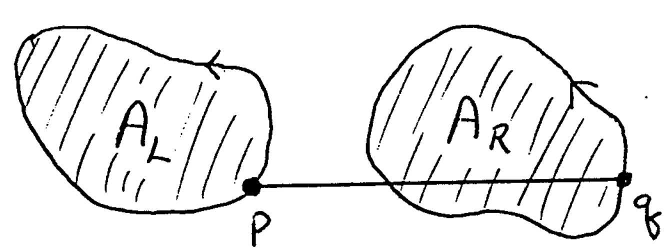
ABSTRACT. A planimeter is a mechanical instrument used to determine the area of a region in the plane. As the boundary of the region is traced, a wheel attached to the instrument partially rolls and partially slides, recording a component of its motion on the plane. The area of the region is a simple function of the net roll of the wheel. We show how the analogue of this instrument works on the sphere and the hyperbolic plane, and then use the results to give a simple proof of one of the Bonnesen isoperimetric inequalities for these surfaces. The rolling of the wheel can be interpreted as parallel translation for a connection in a certain bundle over the surface. The isoperimetric inequality can then be viewed as a statement about the holonomy of this connection.

THE CASE IN $\mathbb{R}^2$

Before doing the general case, we give an informal description of how a planimeter works in the plane, and how its action is related to the isoperimetric inequality. For more details and a brief history of planimeters, see [2].

Let p and q be distinct points in $\mathbb{R}^2$, and consider the segment joining them. We will call p the left endpoint of the segment and q the right endpoint. This orientation determines a "forward" side of the segment, namely, the direction you face when you stand on the segment with p to your left and q to your right. Let N be a forward-pointing unit vector along the segment.





If the segment moves slightly, an infinitesimal oriented area is swept out, given by $dA = \ell N \cdot \frac{dp+dq}{2}$, where ℓ is the length of the segment (which can be variable), and dp and dq are the infinitesimal displacements of the endpoints. If the endpoints of the moving segment trace out curves, the expression for dA can be integrated, yielding the total oriented area swept out. In particular, if the endpoints trace out closed curves, it is easy to show that the oriented area swept out is $A_R - A_L$, where A_R and A_L are the oriented areas enclosed by the right and left endpoints, respectively.

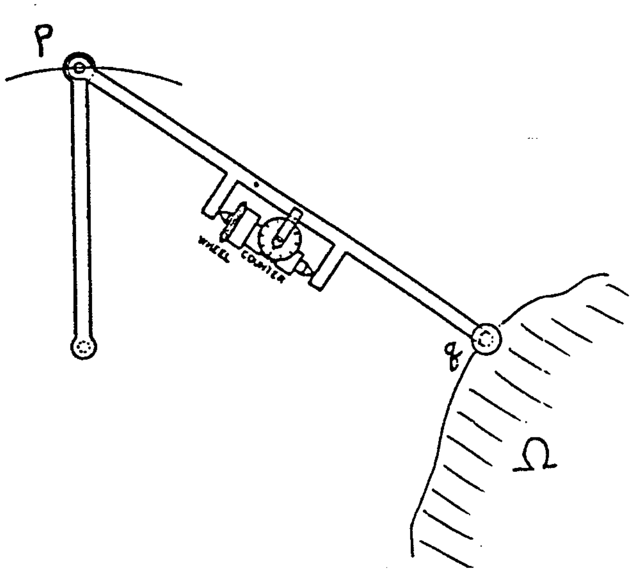
If a wheel is attached to the segment at p with its axis parallel to the segment, it will roll when the segment moves forward but slip when the segment moves sideways. If the segment moves infinitesimally, the "roll" of the wheel will be $d\sigma_L = N \cdot dp$. Similarly, $d\sigma_R = N \cdot dq$ is the infinitesimal roll of a wheel attached to the right endpoint of the segment, and we have $dA = \frac{\ell}{2}(d\sigma_L + d\sigma_R)$.

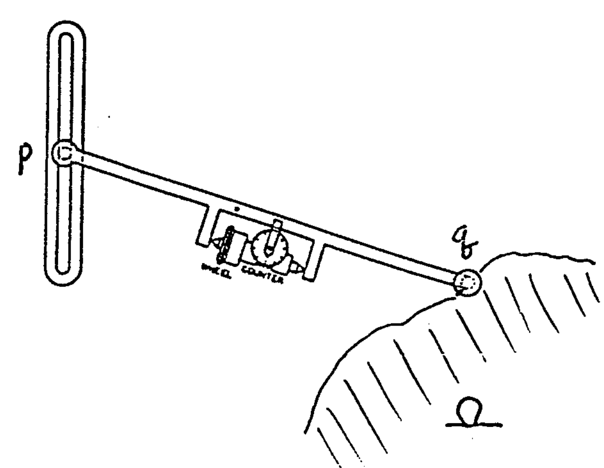
The expression for dA can be written in other ways,

$$(1) \qquad dA \quad = \quad \ell d\sigma_M \quad = \quad \ell d\sigma_L + \frac{\ell^2}{2}\, d\theta \quad = \quad \ell d\sigma_R - \frac{\ell^2}{2}\, d\theta,$$

where $d\sigma_M = \frac{1}{2}(d\sigma_L + d\sigma_R)$ is the infinitesimal roll of a wheel attached to the midpoint of the segment and $d\theta$ is the infinitesimal rotation of the segment.

The moving segment becomes a planimeter when we fix its length ℓ and use it to measure the area of a desired region Ω. This is accomplished by having the right endpoint (the tracer point), move around $\partial\Omega$ in the positive direction (so $A_R = A_\Omega$) while the left endpoint traces the boundary of some region of known area. In typical applications, the path followed by the left endpoint doesn't enclose any area, so $A_L = 0$ (this will always be the case in this paper). The polar and linear planimeters (invented by Jacob Amsler in 1854) are examples of this in which the left endpoint p moves along a circle (not going all the way around) and a line, respectively, but it is important to note that p can move along any curve that doesn't enclose any area.





When the planimeter is used to measure the area of Ω, it mechanically integrates (1). Since it returns to its original position, $d\theta$ integrates to $2\pi n$, where n is the number of rotations made by the planimeter. Thus we have

$$(2) \qquad A_\Omega \;=\; \ell\tilde{\sigma}_M \;=\; \ell\tilde{\sigma}_L + n\pi\ell^2 \;=\; \ell\tilde{\sigma}_R - n\pi\ell^2,$$

where $\tilde{\sigma}_M = \int d\sigma_M$, etc. We call $\tilde{\sigma}_M$ the "total roll" of the wheel. When the planimeter is used in its usual way n is 0. In this case $A_\Omega = \ell\tilde{\sigma}_M = \ell\tilde{\sigma}_L = \ell\tilde{\sigma}_R$, and so the area can be read off of a scale attached to the wheel that incorporates the constant of proportionality ℓ. It's of interest to note that the three wheels roll the same amount, and it's not hard to show that a wheel located at any point along the segment would also.

For isoperimetric inequalities, the interesting case is when the region Ω is measured in such a way that the planimeter arm makes a counterclockwise rotation, that is, $n = 1$. This can be done only if the length ℓ of the planimeter is related to the region in some way. Two situations of interest in this paper are when ℓ is a half-width of Ω or the circumradius of Ω (how this is accomplished will be explained in a moment). When Ω can be measured in this way, the third expression for $A = A_\Omega$ in (2) can be written as $\tilde{\sigma}_R\ell = A + \pi\ell^2$. Note that for each position of the planimeter, the infinitesimal roll $d\sigma_R$ of the wheel at the right endpoint (which traces $\partial\Omega$) is the component of the infinitesimal length ds of $\partial\Omega$ that is perpendicular to the planimeter. Thus $L = \int ds \geq \int d\sigma_R = \tilde{\sigma}_R$, where L is the length of $\partial\Omega$, with equality if and only if the planimeter arm stays perpendicular to $\partial\Omega$. Thus we have $L\ell \geq A + \pi\ell^2$. This inequality is algebraically equivalent to three inequalities that give lower bounds on the isoperimetric defect [6, Lemma 1], two of which are listed on the left below. The same algebraic manipulations applied to $\tilde{\sigma}_R\ell = A + \pi\ell^2$ yield the equalities listed on the right.

$$L^2 - 4\pi A \geq (L - 2\pi\ell)^2 \qquad\qquad \tilde{\sigma}_R^2 - 4\pi A = (\tilde{\sigma}_R - 2\pi\ell)^2$$

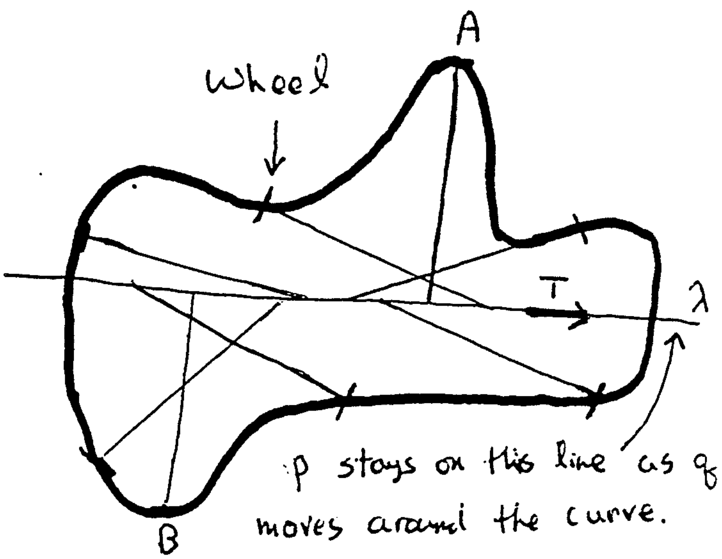
$$L^2 - 4\pi A \geq \frac{1}{\ell^2}(\pi\ell^2 - A)^2 \qquad\qquad = \frac{1}{\ell^2}(\pi\ell^2 - A)^2$$

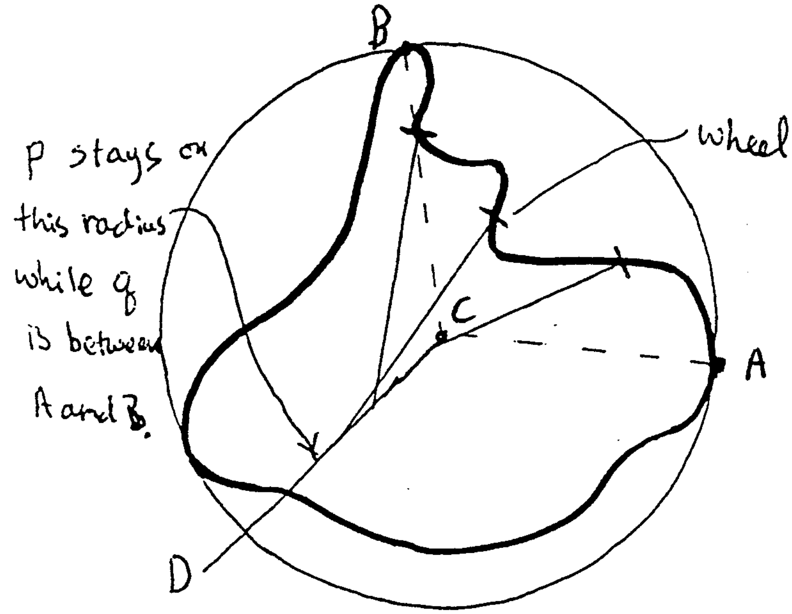
When ℓ is the circumradius R, the second inequality on the left becomes

$$L^2 - 4\pi A \geq \frac{1}{R^2}(\pi R^2 - A)^2.$$

This is known as a Bonnesen inequality since the right hand side vanishes if and only if Ω is a disk of radius R. Thus the isoperimetric inequality $L^2 \geq 4\pi A$ is an equality if and only if the region is a disk. [6]

How to Measure Ω. The motion of the planimeter is given by a pair of piecewise C^1, closed curves γ_L and γ_R that give the motion of the left and right endpoints. The curve γ_R must be a positively oriented parameterization of $\partial\Omega$, the curve γ_L must enclose zero area, the distance between $\gamma_L(t)$ and $\gamma_R(t)$ must be constant, and, for the isoperimetric inequality results, the vector $\gamma_R(t) - \gamma_L(t)$ must rotate counterclockwise once. Evidently a motion of the planimeter is a section of the circle bundle $S^1\mathbb{R}^2 \to \mathbb{R}^2$ along $\partial\Omega$.





The Half-Width Measurement. Let λ be a line such that the maximum distance from λ to points on $\partial\Omega$ is attained by two points A and B on opposite sides of λ. Let ℓ be this maximum distance. We will measure Ω with a planimeter of length ℓ. Let $\gamma_R \colon [0,2] \to \mathbb{R}^2$ be a positively oriented, piecewise C^1 parameterization of $\partial\Omega$ such that $\gamma_R(0) = \gamma_R(2) = A$ and $\gamma_R(1) = B$. Then $\gamma_L(0) = \gamma_L(2)$ and $\gamma_L(1)$ are determined—they are the unique points on λ closest to A and B, respectively. Let T be a unit vector parallel to λ such that the basis $T, \gamma_R(0) - \gamma_L(0)$ is positively oriented. For all other values of $t \in [0,2]$, the circle $C_\ell(\gamma_R(t))$ of radius ℓ centered at $\gamma_R(t)$ meets λ in one or two points. Exactly one point $\gamma_L(t)$ in this intersection is picked out by the condition $\big(\gamma_R(t) - \gamma_L(t)\big) \cdot T \le 0$ for $t \in [0,1]$ and $\big(\gamma_R(t) - \gamma_L(t)\big) \cdot T \ge 0$ for $t \in [1,2]$, and the resulting curve γ_L is easily seen to be C^1. The resulting motion of the planimeter clearly makes one counterclockwise rotation as it measures Ω, and γ_L encloses no area, as required. This construction, up to reparameterization, is the same as the one used in a proof of the isoperimetric inequality due to Schmidt [4, pg. 64].

The Circumradius Measurement. Let D_R be the disk with radius R such that ∂D_R is the circumscribing circle of Ω. We will measure Ω with a planimeter of length R. Let $\gamma_R\colon [0,1] \to \mathbb{R}^2$ be a positively oriented, piecewise C^1 parameterization of $\partial\Omega$ such that $\gamma_R(0) = \gamma_R(1)$ is on ∂D_R. For $\gamma_R(t) \in \partial D_R$, define $\gamma_L(t)$ to be C, the center of D_R.

Suppose that A and B are two "consecutive" points of $\partial\Omega$ on the circumscribing circle, that is, $A = \gamma_R(a)$ and $B = \gamma_R(b)$ are on ∂D_R, for some $a < b$, and $\gamma_R(t) \notin \partial D_R$ for $a < t < b$. Note that the measure of the oriented angle $\angle ACB$ must be less than or equal to π (otherwise ∂D_R would not be the circumscribing circle). Let $\overrightarrow{CD}$ be the ray opposite the angle bisector of $\angle ACB$ (if the measure of $\angle ACB$ is π, $\overrightarrow{CD}$ is taken so that $B - C$, $D - C$ form a positively oriented, orthogonal basis). For each q in the interior of D_R, the intersection of $C_R(q)$ and $\overrightarrow{CD}$ consists of a unique point p, and the mapping $F\colon q \mapsto p$ is smooth. This mapping extends continuously to A and B, which are mapped to C, since the angles $\angle DCA$ and $\angle BCD$ are obtuse or right. We then let $\gamma_L(t) = F(\gamma_R(t))$ for $a \le t \le b$. By construction, γ_L is piecewise C^1. The net rotation of $\gamma_R(t) - \gamma_L(t)$ for $a \le t \le b$ is the same as the measure of $\angle ACB$, and so the resulting motion of the planimeter makes one counterclockwise rotation as it measures Ω. The image of γ_L is a finite or countably infinite union of line segments with common endpoint C, and so γ_L encloses no area.

Constant Curvature Surfaces

The discussion above is made more precise by noting that the configuration space of the planimeter moving in the plane is

$$\mathbb{R}^2_\ell = \{(p,q) \in \mathbb{R}^2 \times \mathbb{R}^2 \colon \; \|p - q\| = \ell\},$$

that the quantities dA, $d\sigma_L$, $d\sigma_M$, $d\sigma_R$, and $d\theta$ are 1-forms on this space, and that the equations (1) express how these 1-forms are related. None of these forms is exact, and their analogs on constant curvature surfaces will be denoted by α, σ_L, σ_M, σ_R, and Θ.

The Setting. To simplify topological considerations, we will consider only the sphere, the Euclidean plane, and the hyperbolic plane. Let M be S^2, $\mathbb{R}^2$, or H^2 with constant curvature k.

Let D_ℓ be a disk of radius ℓ in M. We will need various identities relating the area A_ℓ of D_ℓ, and the length L_ℓ and geodesic curvature c_ℓ of ∂D_ℓ. These quantities are given in the following table.

(3)

	$k>0$	$k=0$	$k<0$
A_ℓ	$2\pi\frac{1-\cos(\sqrt{k}\ell)}{k}$	$\pi\ell^2$	$2\pi\frac{1-\cosh(\sqrt{-k}\ell)}{k}$
L_ℓ	$2\pi\frac{\sin(\sqrt{k}\ell)}{\sqrt{k}}$	$2\pi\ell$	$2\pi\frac{\sinh(\sqrt{-k}\ell)}{\sqrt{-k}}$
c_ℓ	$\sqrt{k}\cot(\sqrt{k}\ell)$	$1/\ell$	$\sqrt{-k}\coth(\sqrt{-k}\ell)$

The most important identity is the isoperimetric *equality* for D_ℓ: $L_\ell^2 = 4\pi A_\ell - kA_\ell^2$. Other identities will be introduced as needed. All are easily verified using (3).

The configuration space of the planimeter of length $\ell > 0$ on M is

$$M_\ell = \{(p,q) \in M\times M \mid d(p,q) = \ell\}.$$

When M is S^2 we require that ℓ be less than $\pi/\sqrt{k}$ (half the circumference) to avoid antipodal points. (On other constant curvature surfaces, one needs to work on orientable subsets, and ℓ needs to be small enough so that no segment of length ℓ contains conjugate points or points in the cut locus of either endpoint.) We will identify the pair (p,q) with the oriented segment joining p to q, and will call p and q the left and right endpoints of the segment, respectively. A motion of the planimeter is a piecewise C^1 curve in M_ℓ.

The tangent space $T_{(p,q)}M_\ell$ consists of $(X,Y) \in T_pM \oplus T_qM$ such that X and Y have the same component in the direction of the segment. More precisely, let $u_p \in T_pM$ be the unit vector such that $\mathrm{Exp}_p(d(p,q)u_p) = q$, and let u_q be the parallel translate of u_p along the segment to q. Then

$$T_{(p,q)}M_\ell = \{(X,Y) \in T_pM \oplus T_qM \mid \langle X, u_p\rangle = \langle Y, u_q\rangle\}.$$

The 1-forms σ_L, σ_M, σ_R, α, and Θ. Before defining σ_L, σ_M, σ_R, α, and Θ on M_ℓ, a few comments are in order. The forms σ_L, σ_R, and σ_M are to measure the infinitesimal roll of wheels attached to the endpoints and midpoint of the segment, and α is to measure the infinitesimal oriented area swept out by the segment. The form Θ is to measure the infinitesimal rotation of the segment as it moves, that is, the deviation of the segment from parallel translation. When the curvature is non-zero, this measure of rotation depends on the point on the segment where it is being measured, and so we will need to define Θ_L and Θ_R for each endpoint.

When the segment moves in the direction of its length, that is, in the direction $(u_p, u_q) \in T_{(p,q)}M_\ell$, the wheels don't roll, no area is swept out, and the segment doesn't rotate. Thus

(u_p, u_q) is in the kernel of all of these forms. Since $\dim M_\ell = 3$, these forms at (p,q) are in a two-dimensional subspace of $T^*_{(p,q)}M_\ell$, and so are linear combinations of any two that are independent.

Let N be the "forward-pointing" unit vector field along the segment, that is, N_p is such that u_p, N_p, form an oriented, orthonormal basis of T_pM; N_x is the parallel translate of N_p along the segment, for every x on the segment. We will generally use the notation u_L, u_R, N_L, and N_R for u_p, u_q, N_p, and N_q to emphasize "left" and "right." We can now define σ_L and σ_R by

$$\sigma_L(X,Y) = \langle X, N_L\rangle \qquad \text{and} \qquad \sigma_R(X,Y) = \langle Y, N_R\rangle\,.$$

Fix p and let D_ℓ be the disk of radius ℓ centered at p. Since M is isotropic, the quantities $A_\ell/2\pi$ and $L_\ell/2\pi$ are the rates at which the area of D_ℓ and arc-length of ∂D_ℓ are swept out as a radial segment of D_ℓ rotates about p in the positive direction. Thus we should have

$$\alpha(0,Y) = \frac{A_\ell}{2\pi}\Theta_L(0,Y) \qquad \text{and} \qquad \sigma_R(0,Y) = \frac{L_\ell}{2\pi}\Theta_L(0,Y).$$

In addition, as the segment rotates, the rate of rotation as measured at the right endpoint should be given in terms of the curvature as

$$\Theta_R(0,Y) = c_\ell\sigma_R(0,Y).$$

Similarly,

$$\alpha(X,0) = -\frac{A_\ell}{2\pi}\Theta_R(X,0), \quad \sigma_R(X,0) = -\frac{L_\ell}{2\pi}\Theta_R(X,0), \quad \text{and} \quad \Theta_L(X,0) = -c_\ell\sigma_L(X,0).$$

These requirements force α, Θ_L, Θ_R to be the following linear combinations of σ_L and σ_R, which we take as definitions:

$$(4) \qquad \alpha = \frac{A_\ell}{L_\ell}(\sigma_L + \sigma_R), \qquad \Theta_L = -c_\ell\sigma_L + \frac{2\pi}{L_\ell}\sigma_R, \qquad \text{and} \qquad \Theta_R = -\frac{2\pi}{L_\ell}\sigma_L + c_\ell\sigma_R.$$

Note that when M is the Euclidean plane, $\Theta_L = \Theta_R$, since $c_\ell = 2\pi/L_\ell = 1/\ell$ in this case. In fact, $\Theta_L = \Theta_R = d\theta$, where θ satisfies $q - p = \ell(\cos\theta\, e_1 + \sin\theta\, e_2)$, and e_1, e_2 form the standard basis of $\mathbb{R}^2$.

Finally, the 1-form σ_M should measure the infinitesimal roll of the wheel at the midpoint of the segment, namely $\sigma_M(X,Y) = \langle N_M, Z_M\rangle$, where N_M is the forward-pointing normal at the midpoint, and Z_M is the infinitesimal displacement of the midpoint corresponding to the infinitesimal displacement (X,Y) of (p,q). Since the motion of the segment is a variation of a geodesic, the displacement field Z along the segment is the Jacobi field with values X and Y at the endpoints. At the midpoint, the normal component of Z_M is $\langle N_M, Z_M\rangle = \frac{L_{\ell/2}}{L_\ell}\big(\langle N_L, X\rangle + \langle N_R, Y\rangle\big)$, and so

$$\sigma_M = \frac{L_{\ell/2}}{L_\ell}(\sigma_L + \sigma_R). \tag{4'}$$

The main property of the form α is that if $\Gamma = (\gamma_L, \gamma_R)$ is a closed, piecewise C^1 curve in M_ℓ, then $\int_\Gamma \alpha = A_R - A_L$, where A_L and A_R are, respectively, the oriented areas enclosed by the curves γ_L and γ_R in M.

This is true, and easier to prove, in a more general setting. Let M_{seg} be the space of oriented segments in M, that is, the set of all $(p, q) \in M\times M$ such that there is a unique geodesic segment joining p to q. (This is $M\times M$ except when M is S^2, in which case antipodal pairs are omitted.) Thus $M_\ell \subset M_{\text{seg}}$. Given $(p, q) \in M_{\text{seg}}$ with p and q distinct, the vectors u_L, u_R, N_L, and N_R can be defined as above, which allow the definitions of σ_L, σ_R, α, Θ_L, and Θ_R to be extended to M_{seg} off the diagonal. In particular we have

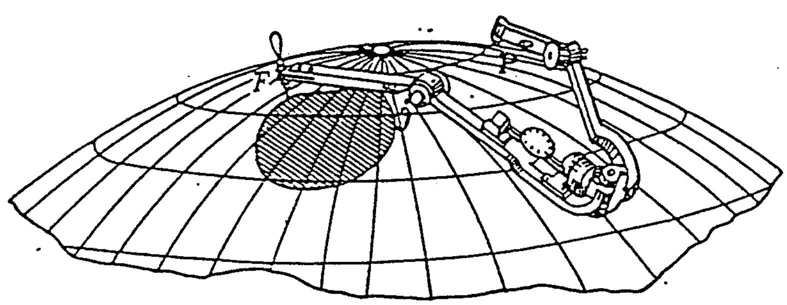
$$\alpha(X, Y) = \frac{A_r}{2\pi}(\Theta_L(0, Y) - \Theta_R(X, 0)),$$

where $r\colon M_{\text{seg}} \to \mathbb{R}$ measures the length of the segment, *i.e.*,, $r(p, q)$ is the distance between p and q. Note that α extends to the diagonal since $A_0 = 0$. For α on M_{seg} we have the following.

Lemma 1. *Let $\Gamma = (\gamma_L, \gamma_R)$ be a closed, piecewise C^1 curve in M_{seg}. Then $\int_\Gamma \alpha = A_R - A_L$, where A_L and A_R are, respectively, the oriented areas enclosed by γ_L and γ_R.*

This is shown by proving that $d\alpha = dA_R - dA_L$.

How a Planimeter Works on M.



The equations (4, 4′) along with the identities $\pi A_\ell = L_{\ell/2}^2$ and $2\pi A_\ell + c_\ell L_\ell A_\ell = L_\ell^2$ yield the following formulas for α:

$$
\begin{aligned}
\alpha &= \frac{L_{\ell/2}}{\pi}\sigma_M = \frac{L_\ell}{2\pi}\sigma_L + \frac{A_\ell}{2\pi}\Theta_L \\
&= \frac{L_\ell}{2\pi}\sigma_R - \frac{A_\ell}{2\pi}\Theta_R = \frac{1}{c_\ell}\sigma_R - \frac{A_\ell}{L_\ell c_\ell}\Theta_L.
\end{aligned} \tag{5}
$$

These are the analogs of (1). Note that we need to use caution when using the last formula on the sphere, as $c_\ell = 0$ when $\ell = \pi\sqrt{k}/2$ (one-fourth of the circumference).

Let $\Omega \subset M$ be simply connected with piecewise C^1 boundary. Suppose Ω can be measured with a planimeter of length ℓ, that is, there is a closed, piecewise C^1 curve $\Gamma = (\gamma_L, \gamma_R)$ in M_ℓ such that γ_L traverses $\partial\Omega$ in the positive direction and γ_R encloses no area. As before, we have $\int_\Gamma \alpha = A_\Omega$ and $\int_\Gamma \sigma_M = \tilde{\sigma}_M$, etc. Integrating Θ_L and Θ_R is a bit more involved.

Lemma 2. *$\int_\Gamma \Theta_L = 2\pi n$ and $\int_\Gamma \Theta_R = 2\pi n - kA_\Omega$, where n is the number of rotations made by the planimeter.*

Remark. If M is $\mathbb{R}^2$ or H^2, the number of rotations made by the planimeter is unambiguous —the rotation can be measured relative to some global frame on M. When M is the sphere, we will assume that there is some point x_0 such that x_0 is not in the closure of Ω or on the segment $\Gamma(t)$ for any t (this will be the case, for example, if Ω is contained in a hemisphere and $\ell < \pi\sqrt{k}/2$). Then the rotation can be measured relative to some frame on $M \setminus \{x_0\}$.

Proof. Let $S^1M \to M$ be the unit circle bundle of M, and note that the maps $f_L, f_R \colon M_\ell \to S^1M$ given by $f_L(p,q) = N_p$ $(= N_L)$ and $f_R(p,q) = N_q$ $(= N_R)$ are diffeomorphisms. Let Θ be the rotation form on S^1M, *i.e.*, the reduction of the Riemannian connection form on the frame bundle $F(M) \to M$ to the circle bundle. We have $f_L^*(\Theta) = \Theta_L$ and $f_R^*(\Theta) = \Theta_R$. Thus $\int_\Gamma \Theta_L = \int_{f_L \circ \Gamma} \Theta$ and $\int_\Gamma \Theta_R = \int_{f_R \circ \Gamma} \Theta$.

Let $\tilde{\Omega} \subset M$ be simply connected with $\partial\tilde{\Omega}$ traversed in the positive direction by the closed, piecewise C^1 curve $\tilde{\gamma} : [0,1] \to M$. When M is the sphere, assume the closure of $\tilde{\Omega}$ is not all of M. Suppose $\tilde{\Gamma}$ is a section of S^1M along $\tilde{\gamma}$. Since Θ is the connection form on S^1M, we have $d\Theta = -k\,dA$. It follows that $\int_{\tilde{\Gamma}} \Theta = \Delta\theta - kA_{\tilde{\Omega}}$, where $\Delta\theta$ is the net rotation of the section $\tilde{\Gamma}$, that is, the angle from $\tilde{\Gamma}(0)$ to $\tilde{\Gamma}(1)$.

The result follows since γ_R encloses Ω and γ_L doesn't enclose any area. □

In its typical use, the planimeter doesn't make a full rotation ($n = 0$). The four expressions in (5) then yield

$$
A_\Omega = \frac{L_{\ell/2}}{\pi}\tilde{\sigma}_M = \frac{L_\ell}{2\pi}\tilde{\sigma}_L = \frac{L_\ell}{2\pi}\tilde{\sigma}_R + \frac{A_\ell}{2\pi}kA_\Omega = \frac{1}{c_\ell}\tilde{\sigma}_R.
$$

The last two expressions agree by the identity $L_\ell c_\ell = 2\pi - kA_\ell$. We see that the roll of each wheel is proportional to the area of Ω, although unlike the flat case, the constant of proportionality depends on the location of the wheel. In fact, it can be shown that if $\tilde{\sigma}$ is the total roll of a wheel at some point on the planimeter, there is a constant C depending only on ℓ and the location of the wheel such that $A_\Omega = C\tilde{\sigma}$ (this still assumes $n = 0$).

On the sphere when $\ell = \pi\sqrt{k}/2$ we have $c_\ell = 0$ and the last formula in (5) reduces to $0 = \sigma_R - \frac{1}{\sqrt{k}}\Theta_L$, which integrates to $\tilde{\sigma}_R = 0$. Thus in this case the roll of the wheel at the right endpoint is useless in determining the area of the region!

Some of this theory for a planimeter on a sphere was certainly known to Amsler. Henrici [3, pg. 513] reports that Amsler published a paper [1] describing a planimeter to be used to measure areas on a globe. I have not been able to obtain a copy of this, however de Morin [5, pp. 75–80] works out a version of (5) for α in terms of the infinitesimal roll σ at an arbitrary point on the planimeter and the infinitesimal rotation Θ of the planimeter at the wheel. Unfortunately, he was not aware of Lemma 2, and took $\int_\Gamma \Theta$ to be zero (he considered only the case of no net rotation). Thus his integrated formula is valid only when the wheel is at the left endpoint of the segment. Interestingly, that's where the wheel is in the picture in his book, which is reproduced at the beginning of this subsection.

Isoperimetric Inequalities. The isoperimetric inequality for Ω is $L^2 \geq 4\pi A - kA^2$, where $A = A_\Omega$ and L is the length of $\partial\Omega$. [6] We obtained this by measuring Ω with a planimeter of length ℓ that allows the planimeter to make a counterclockwise rotation ($n = 1$). In this case, the integrated form of the third expression for α in (5) is $A = \frac{L_\ell}{2\pi}\tilde{\sigma}_R + \frac{A_\ell}{2\pi}kA - A_\ell$, which can be written as $\tilde{\sigma}_R L_\ell = 2\pi(A + A_\ell) - kAA_\ell$ (compare with $\tilde{\sigma}_R \ell = A + \pi\ell^2$ for the flat case). As before we have $L \geq \tilde{\sigma}_R$, since σ_R records only the component of the length of $\partial\Omega$ in the direction of N_R. Thus $LL_\ell \geq 2\pi(A + A_\ell) - kAA_\ell$. This inequality is algebraically equivalent to three inequalities that give lower bounds on the isoperimetric defect [6, Lemma 6], listed on the left below. The same algebraic manipulations applied to $\tilde{\sigma}_R L_\ell = 2\pi(A + A_\ell) - kAA_\ell$ yield the equalities listed on the right.

$$L^2 - 4\pi A + kA^2 \geq \left(L - \frac{L_\ell}{A_\ell}A\right)^2$$
$$L^2 - 4\pi A + kA^2 \geq (L - L_\ell)^2 + k(A - A_\ell)^2$$
$$L^2 - 4\pi A + kA^2 \geq \left(\frac{L_\ell}{2\pi}\right)^{-2}(A_\ell - A)^2$$

$$\begin{aligned}\tilde{\sigma}_R^2 - 4\pi A + kA^2 &= \left(\tilde{\sigma}_R - \frac{L_\ell}{A_\ell}A\right)^2 \\ &= (\tilde{\sigma}_R - L_\ell)^2 + k(A - A_\ell)^2 \\ &= \left(\frac{L_\ell}{2\pi}\right)^{-2}(A_\ell - A)^2\end{aligned}$$

When ℓ is the circumradius R, the last inequality on the left is a Bonnesen inequality as before, since the right hand side vanishes if and only if Ω is a disk of radius R. Thus $L^2 \geq 4\pi A - kA^2$ is an equality if and only if the region is a disk.

How to Measure Ω. The constructions for the half-width and circumradius measurements in the Euclidean case are valid in the Hyperbolic case with just minor changes. The unit vector T parallel to λ in the half-width construction is replaced by a unit vector field T tangent to λ. In both constructions the vector $\gamma_R(t) - \gamma_L(t)$ is replaced by the vector $u_L(\Gamma(t))$. Then, for example, in the half-width construction $\gamma_L(t)$ is on λ, so it makes sense to compute $\langle u_L(\Gamma(t)), T(\gamma_L(t))\rangle$. The vectors $A-C$, $B-C$, and $D-C$ in the circumradius construction must be replaced by vectors in $T_C M$ tangent to the segments CA, CB, and CD.

On the sphere one needs to be more careful. The half-width construction is valid (with the same modifications as above) if the points A and B, farthest from the geodesic λ, are not antipodal points. The circumradius construction is valid if $R < \pi\sqrt{k}/2$, that is, when Ω is contained in a hemisphere, and when the ray $\overrightarrow{CD}$ is replaced by a geodesic segment CDC', where C' is the point antipodal to C.

The Isoperimetric Inequality and Holonomy

There is a sense in which the motion of the wheel can be thought of as parallel translation for a connection on an affine line bundle over M_ℓ. The equations $\tilde{\sigma}_R \ell = A + \pi\ell^2$, in the flat case, and $\tilde{\sigma}_R L_\ell = 2\pi(A + A_\ell) - kAA_\ell$, in the curved case, and their algebraic equivalents are statements about the holonomy of this connection. This holonomy captures the essence of the isoperimetric inequality, since the inequality boils down to the trivial observation that $L \geq \tilde{\sigma}_R$.

References

1. J. Amsler, *Neuere Planimeter-Constructionen*, Zeitschrift für Instrumentenkunde (1884).
2. R. L. Foote, *A Plenitude of Planimeters*, in preparation.
3. O. Henrici, *Report on Planimeters*, British Assoc. for the Advancement of Science, Report of the 64th meeting (1894), 496–523.
4. R. Milliman and G. Parker, *Elements of Differential Geometry*, Prentice-Hall, Englewood Cliffs, New Jersey, 1977.
5. H. de Morin, *Les Appareils D'Intégration*, Gauthier-Villars, Paris, 1913.
6. R. Osserman, *Bonnesen-style isoperimetric inequalities*, MAA Monthly **86** (1979), 1–29.

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