The Curl Operator in Two Dimensions
Fall 2009

Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on a subset of $\mathbb{R}^2$ with differentiable coefficients. The curl of $\mathbf{F}$ is the scalar function defined by

$$\operatorname{curl}\mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Here are three alternate formulas for you to ponder.

$$\operatorname{curl}\mathbf{F} = \det\begin{pmatrix} \partial/\partial x & \partial/\partial y \\ P & Q \end{pmatrix},$$

$$\operatorname{curl}\mathbf{F} = -\operatorname{div}\mathbf{F}_\perp = -\nabla\cdot\mathbf{F}_\perp,$$

and

$$\operatorname{curl}\mathbf{F} = \nabla_\perp\cdot\mathbf{F},$$

where $\nabla_\perp = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)_\perp = -\frac{\partial}{\partial y}\mathbf{i} + \frac{\partial}{\partial x}\mathbf{j}$.

The curl of $\mathbf{F}$ is a measure of the local rate of rotation of the flow of $\mathbf{F}$. More specifically, if P is a point, then $\frac{1}{2}\operatorname{curl}\mathbf{F}(P)$ is the average rate of rotation of small region centered at P as it follows the flow of $\mathbf{F}$ measured in radians per unit time. If $\operatorname{curl}\mathbf{F}(P) > 0$ the rotation is counterclockwise; if $\operatorname{curl}\mathbf{F}(P) < 0$ the rotation is clockwise. A motivation of this fact, including the factor of $1/2$, can be seen by taking the curl of the standard rotation field. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ be the position vector from the origin. The standard rotation field is $\mathbf{r}_\perp = -y\mathbf{i} + x\mathbf{j}$. The flow curves of $\mathbf{F}$ are easily seen to be $\gamma(t) = (r_0\cos t, r_0\sin t)$, where $r_0 > 0$ is a constant. Thus, the flow of $\mathbf{F}$ is the entire xy-plane rotating about the origin with a rotational velocity of 1 radian per unit time in the counterclockwise direction. We have $\operatorname{curl}\mathbf{F} = 2$, and so $\frac{1}{2}\operatorname{curl}\mathbf{F}$ gives the rate of rotation of the flow of $\mathbf{F}$.