

Homogeneous Complex Monge–Ampère Equations and Algebraic Embeddings of Parabolic Manifolds

ROBERT L. FOOTE

ABSTRACT. Let M^n be a Stein manifold, and let $\tau : M \rightarrow [0, R^2)$, $0 < R \leq \infty$, be a strictly plurisubharmonic exhaustion. The pair (M, τ) is said to be *strictly parabolic at infinity* if $u = \log \tau$ satisfies the Monge–Ampère equation $(\partial\bar{\partial}u)^n \equiv 0$ outside some compact set K . Additionally, (M, τ) is of *Reinhardt type* if τ can be written locally as a Reinhardt function on $M - K$. The set $M - K$ admits a Monge–Ampère foliation in a well-known way. With an assumption on the holomorphic twist of this foliation when $R = \infty$ we prove a curvature estimate, which, via a theorem due to Demailly, implies that M embeds algebraically into \mathbb{C}^{2n+1} . When $R \leq \infty$ and $K = \tau^{-1}(0)$ we prove, under a weaker regularity assumption than usual, that M is biholomorphic to the ball in \mathbb{C}^n of radius R .

1. Introduction.

Outline. Let M be a connected, complex manifold of dimension n , and let $\tau : M \rightarrow [0, R^2)$, $0 < R \leq \infty$, be a strictly plurisubharmonic (*sps**h*) exhaustion. Stoll [S2] defined a *strictly parabolic manifold* to be one in which the function $u = \log \tau$ satisfies the homogeneous Monge–Ampère equation $(\partial\bar{\partial}u)^n = 0$ on $M - \tau^{-1}(0)$. He proved that if τ is C^∞ , the only example is the R -ball in \mathbb{C}^n with the standard exhaustion $\tau_0(z) = |z|^2$. Burns [B1] gave an alternate proof of this and, using a curvature estimate, drew the same conclusion when τ is C^5 and $R = \infty$. It is unknown if the conclusion holds when τ is C^5 and $R < \infty$.

In this paper we follow a generalization of the notion of strictly parabolic manifold due to Burns [B2], [F2]. Let M , τ , and u be as above with $R = \infty$, except that $(\partial\bar{\partial}u)^n = 0$ is required to hold only outside some compact subset of M . We will say that M , together with τ , is *strictly parabolic at infinity*.

Burns conjectured that a Stein manifold M is biholomorphic to an algebraic submanifold of \mathbb{C}^N if and only if M admits an exhaustion τ that makes M strictly parabolic at infinity. We prove a special case of this conjecture.

It is well-known that the local conditions imposed on τ and u , namely

$$(1.1) \quad \tau \text{ is } spsh, \quad u = \log \tau, \quad \text{and} \quad (\partial\bar{\partial}u)^n \equiv 0,$$

give rise to a foliation by Riemann surfaces, known as the *Monge-Ampère foliation*. Section 2 contains a review and further study of the local geometry of this foliation.

In Section 3 we introduce an additional local condition on τ . We say that τ is *locally Reinhardt* if local holomorphic coordinates can be chosen so that τ is a Reinhardt function. In a related coordinate system, it follows that τ satisfies the real analogue of (1.1). There is a real Monge-Ampère foliation, and some results of the author [F3] can be used. As an application, we prove a version of Stoll's theorem that is valid when τ is C^5 and $R < \infty$.

The main results are in Section 4. In Theorem 4.4 we assume that M is strictly parabolic at infinity and satisfies a curvature estimate. Using a theorem due to Demailly [D], we prove that M embeds algebraically into \mathbb{C}^{2n+1} . Theorem 4.5 is a special case of Burns' conjecture. Given M that is strictly parabolic at infinity, we assume the Reinhardt condition and an assumption about the twist of the foliation. We show that the curvature estimate of Theorem 4.4 is satisfied and conclude that M admits an algebraic embedding. We give an example showing that the conditions in (1.1) are, by themselves, insufficient to prove the curvature estimate—some condition governing the twist of the foliation is necessary.

Portions of this paper originally appeared as parts of my doctoral dissertation [F1], which was written under the direction of Daniel M. Burns. I would like to take this opportunity to thank him for the many conversations we had, both mathematical and non-mathematical, while I was his student.

Notational Conventions. The complexified tangent bundle of M will be denoted by $T^{\mathbb{C}}M$. If $X \in T^{\mathbb{C}}M$, then $X^{1,0}$ and $X^{0,1}$ will denote its components in $T^{1,0}M$ and $T^{0,1}M$.

Let $f : M \rightarrow \mathbb{R}$ be C^2 . The complex Hessian of f will be denoted by $H_{\mathbb{C}}(f)$ ($= \sum \partial^2 f / \partial z^\alpha \partial \bar{z}^\beta dz^\alpha \otimes d\bar{z}^\beta$ in local coordinates). Since τ is *spsh*, $H_{\mathbb{C}}(\tau)$ determines a Hermitian metric on $T^{1,0}M : (Z, W) \mapsto H_{\mathbb{C}}(\tau)(Z, \bar{W})$, which will be called the τ -metric. Its Kähler form is $\frac{i}{2} \partial\bar{\partial}\tau = dd^c\tau$, where $d^c = \frac{i}{4}(\bar{\partial} - \partial)$. Note that $\partial\bar{\partial}\tau(Z, \bar{W}) = H_{\mathbb{C}}(\tau)(Z, \bar{W})$ for $Z, W \in T^{1,0}M$, but that $\partial\bar{\partial}\tau(X, Y) = H_{\mathbb{C}}(\tau)(X, Y) - H_{\mathbb{C}}(\tau)(Y, X)$ for general $X, Y \in T^{\mathbb{C}}M$. The volume form is given by $dV = (dd^c\tau)^n$.

A typical level set of τ (hence of $u = \log \tau$) will be denoted by Σ . Let $T^{1,0}\Sigma = T^{1,0}M \cap \ker \partial\tau$, $T^{0,1}\Sigma = \overline{T^{1,0}\Sigma}$, $H\Sigma = T\Sigma \cap J T\Sigma$, and $H^{\mathbb{C}}\Sigma = H\Sigma \otimes \mathbb{C} = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$.

An additional notational convention concerning indices is made in Remark 2.1.

2. Local Geometry of the Monge-Ampère Foliation.

Let M be a complex manifold of dimension n . Let $u : M \rightarrow \mathbb{R}$ be a solution of the homogeneous Monge-Ampère equation and side conditions

$$(2.1) \quad (\partial\bar{\partial}u)^n = 0, \quad u = \log \tau, \text{ and } \tau \text{ is spsh.}$$

We assume τ and u are C^5 so that the curvatures of the τ -metric will be C^1 .

The local geometric consequences of (2.1) have been widely studied, e.g., [BB], [BK], [B1–2], [F1–3], [Fr], [L1–2], [P1–4], [PW1–2], [S1–3], [W1–4]. In particular, we have the following (see [BK], [S2], [B1], [BB]):

- (1) $\text{rank}(\partial\bar{\partial}u) = \text{rank}H_{\mathbb{C}}(u) = n - 1$.
- (2) M is foliated by complex curves such that u is harmonic on the leaves. A vector $Z \in T^{1,0}M$ is tangent to a leaf if and only if $H_{\mathbb{C}}(u)(Z, \cdot) = 0$.
- (3) The level sets of u (and τ) are non-degenerate, strictly pseudoconvex, and transverse to the leaves.
- (4) $H_{\mathbb{C}}(u)$ is positive definite on the level sets and u is plurisubharmonic (psh).

Remark 2.1. We will adopt the following convention for computations: the indices $1, \dots, n$ will be used with 1 often reserved for the leaf direction. The indices $\alpha, \beta, \gamma, \dots$ will range over $1, \dots, n$; the indices i, j, k, \dots will range over $2, \dots, n$.

Definition 2.2 (cf. [S2], [B1]). Let ξ be the $(1,0)$ -vector field that is $H_{\mathbb{C}}(\tau)$ -dual to $\bar{\partial}\tau$, i.e., ξ satisfies $\bar{\partial}\tau(\bar{Z}) = H_{\mathbb{C}}(\tau)(\xi, \bar{Z}) = \partial\bar{\partial}\tau(\xi, \bar{Z})$ for all $Z \in T^{1,0}$.

For a proof of the next proposition, see [S2], [B1], and [W1].

Proposition 2.3 (Properties of ξ).

- (1) ξ is orthogonal to $T^{1,0}\Sigma$, where Σ is a level set of u and τ .
- (2) ξ is tangent to the foliation.
- (3) $H_{\mathbb{C}}(\tau)(\xi, \bar{\xi}) = \tau$, $\xi\tau = \tau$, and $\xi u = 1$.
- (4) $[\xi, \bar{\xi}] = 0$, so ξ is holomorphic along the leaves of the foliation.
- (5) ξ is completely determined as a $(1,0)$ -vector by (1) and (2) or by (2) and (3).
- (6) ξ is holomorphic if and only if the foliation is holomorphic.

Examples 2.4 (Also see Remark 2.7.).¹

- (1) (Standard Example [S2], [B1])

$$\begin{aligned}\widetilde{M} &= B_R^n = \{z \in \mathbb{C}^n : |z| < R\}, \quad 0 < R \leq \infty, \\ M &= \widetilde{M} - \{0\}, \\ \tau_0(z) &= |z|^2.\end{aligned}$$

The leaves are the complex lines through 0, and ξ is the complex Euler field $\sum z^\alpha \partial / \partial z^\alpha$.

- (2)
$$\begin{aligned}\widetilde{M} &= (\mathbb{C}^*)^n, \\ M &= \widetilde{M} - \{z : |z^\alpha| = 1, \text{ some } \alpha\}, \\ u(z) &= \log \tau(z) = (\sum (\log |z^\alpha|)^2)^{1/2}.\end{aligned}$$

Then $\xi = \frac{2}{u} \sum z^\alpha \log |z^\alpha| \partial / \partial z^\alpha$. When $n = 2$, the leaves are the complex curves contained in the hypersurfaces of the form $\{(z^1, z^2) : |z^1|^r = |z^2|^s\}$. This example is a special case of (4) after a change of coordinates. (See [B2].)

- (3)
$$\begin{aligned}\widetilde{M} &= \{z \in \mathbb{C}^{n+1} : \sum |z^\alpha|^2 = 1\}, \\ M &= \widetilde{M} - \{z : |z| = 1\}, \\ u(z) &= \log \tau(z) = \cosh^{-1}(|z|^2).\end{aligned}$$

The image of $\mathbb{C} \rightarrow \mathbb{C}^{n+1}$, $\rho \mapsto (\cosh \rho, i \sinh \rho, 0, \dots, 0) = (z^1, z^2, 0, \dots, 0)$, is a leaf, and along this leaf $\xi = -iz^2(\partial / \partial z^1) + iz^1(\partial / \partial z^2)$.

- (4) Let $U \subset \mathbb{R}^n$ be open. Let u be a solution of the real Monge-Ampère equation $\text{rank } H(u) = n - 1$, where $H(u) = \sum (\partial^2 u / \partial x^\alpha \partial x^\beta) dx^\alpha \otimes dx^\beta$ is the real Hessian of u . Assume that $e^{u(x)}$ is strictly convex. Let $M = U \times \mathbb{R}^n$ with coordinates (x, y) , $z = x + iy$, and define $u : M \rightarrow \mathbb{R}$ by $u(z) = u(x)$. Since $\partial^2 u / \partial z^\alpha \partial \bar{z}^\beta = \frac{1}{4} \partial^2 u / \partial x^\alpha \partial x^\beta$, the function $u(z)$ satisfies $(\partial \bar{\partial} u)^n \equiv 0$. The strict convexity of $\tau(x) = e^{u(x)}$ implies that $\tau(z) = e^{u(z)}$ is *sps*. If $\frac{\partial}{\partial u}$ denotes the vector field associated with the real Monge-Ampère foliation on U , then $\xi = \frac{\partial}{\partial u} - iJ \frac{\partial}{\partial u}$. The leaves of the real foliation are straight lines along which $\frac{\partial}{\partial u}$ is parallel. It follows that ξ is parallel along the leaves of the complex foliation. If ℓ is a real leaf, the complex leaf containing it is the unique complex line containing ℓ . (See [F3].) The exhaustions on tube domains studied by Patrizio [P3] are of this form. In Section 3 we turn our attention to solutions of (2.1) that can be written in this form locally.

¹ Note: In later sections M will be an open subset of a larger manifold \widetilde{M} , and so this notation is used in these examples.

This can be done in (1) and (2), but not in (3). (In (1) and (2) locally set $\zeta^\alpha = \log z^\alpha$. In (1), if some $z^\alpha = 0$, first make a unitary rotation to a point where all $z^\alpha \neq 0$. For (3), see Remark 3.3.)

- (5) Let $\widetilde{M} \subset \mathbb{C}^N$ be algebraic, and suppose the projection $\pi : \mathbb{C}^N \rightarrow \mathbb{C}^n$ is generic when restricted to \widetilde{M} . Let $\tau = \tau_0 \circ \pi$, where τ_0 is as in (1). Let $M = \widetilde{M} - (\{0\} \cup \{(\partial\bar{\partial}\tau)^n = 0\})$. On M , π is a local biholomorphism. Thus, after a local change of coordinates, this can also be put into the form of (4). (See [B2].)

From Proposition 2.3(3) we have that $du(\operatorname{Re}\xi) = \frac{1}{2}(\partial u + \bar{\partial} u)(\xi + \bar{\xi}) = 1$, and so we denote $\operatorname{Re}\xi$ by $\frac{\partial}{\partial u}$. The vector fields $\frac{\partial}{\partial u}$ and $J\frac{\partial}{\partial u}$ are tangent to the foliation. Since $du(J\frac{\partial}{\partial u}) = -\operatorname{Im}(\xi u) = 0$, $\frac{\partial}{\partial u}$ is orthogonal to the level sets, and the integral curves of $J\frac{\partial}{\partial u}$ are the curves of intersection of the level sets and the leaves of the foliation.

Definition 2.5. Let $S = \bar{\partial}\xi + \partial\bar{\xi}$.

Since ξ is a section of $T^{1,0}M$, $\bar{\partial}\xi$ is a $(0,1)$ -form with coefficients in $T^{1,0}M$. We will think of S as a vector bundle map $T^{\mathbb{C}}M \rightarrow T^{\mathbb{C}}M$. If Z is a $(1,0)$ -vector field on M , then $SZ = [Z, \bar{\xi}]^{0,1}$. Note that S “switches types,” i.e., S maps $T^{1,0}M \rightarrow T^{0,1}M$ and $T^{0,1}M \rightarrow T^{1,0}M$. Also S is real, i.e., S maps $TM \rightarrow TM$. Since $S \equiv 0$ if and only if ξ is holomorphic, S measures the holomorphic “twist” of the foliation. The tensor $(Z, W) \mapsto H_{\mathbb{C}}(\tau)(Z, SW)$ is the twist tensor of Bedford and Burns ([BB], [B1]).

Proposition 2.6 (Properties of S).

- (1) $S\xi = S\bar{\xi} = S\frac{\partial}{\partial u} = SJ\frac{\partial}{\partial u} = 0$.
- (2) S takes its values in $H^{\mathbb{C}}\Sigma$.
- (3) $H_{\mathbb{C}}(\tau)(Z, SW) = H_{\mathbb{C}}(\tau)(W, SZ)$ for all $Z, W \in T^{\mathbb{C}}M$. It follows that $S : H\Sigma \rightarrow H\Sigma$ is symmetric with respect to the associated Riemannian metric, and thus has $2(n-1)$ real eigenvalues.
- (4) $JS = -SJ$.
- (5) If $SX = \lambda X$, then $S(JX) = -\lambda JX$.

Proof.

- (1) This follows from Proposition 2.3(4).

- (2) Let Z be a $(1,0)$ -vector field on M . Then by Proposition 2.3(2,3) we have

$$\begin{aligned}\bar{\partial}u(SZ) &= \bar{\partial}u(\partial\bar{\xi}(Z)) = \bar{\partial}u([Z, \bar{\xi}]) \\ &= -\partial\bar{\partial}u(Z, \bar{\xi}) + Z\bar{\partial}u(\bar{\xi}) - \bar{\xi}\bar{\partial}u(Z) = 0.\end{aligned}$$

Thus $SZ \in T^{0,1}\Sigma$.

- (3) It suffices to show this when Z and W are $(1,0)$ -vector fields on M . The Jacobi identity implies that $S[W, Z] = [SW, Z]^{0,1} + [W, SZ]^{0,1}$. From part (2) we have

$$\begin{aligned}H_{\mathbb{C}}(\tau)(Z, SW) - H_{\mathbb{C}}(\tau)(W, SZ) \\ &= \partial\bar{\partial}\tau(Z, SW) - \partial\bar{\partial}\tau(W, SZ) \\ &= Z\bar{\partial}\tau(SW) - (SW)\bar{\partial}\tau(Z) - \bar{\partial}\tau([Z, SW]) - W\bar{\partial}\tau(SZ) \\ &\quad + (SZ)\bar{\partial}\tau(W) + \bar{\partial}\tau([W, SZ]) \\ &= \bar{\partial}\tau(S[W, Z]) = 0.\end{aligned}$$

- (4) Since S “switches types,” if $Z \in T^{1,0}M$, then $SJZ = iSZ = -JSZ$.

- (5) Suppose $SX = \lambda X$. Then $SJX = -JSX = -\lambda JX$. \square

As a consequence of this proposition, we will usually think of S as a map $H^{\mathbb{C}}\Sigma \rightarrow H^{\mathbb{C}}\Sigma$. Although the eigenvectors of S are necessarily real (for non-zero eigenvalues), note that if $X \in H\Sigma$ is an eigenvector for S with eigenvalue λ , then any complex linear combination of X and JX is an eigenvector of S^2 with eigenvalue λ^2 .

Remark 2.7. The eigenvalues of S in Examples 2.4(1–4) can be shown to be as follows.

- (1) In the standard example the eigenvalues vanish identically since ξ is holomorphic.
- (2) The eigenvalues are all of the form $\lambda = \pm \frac{1}{u}$.
- (3) The eigenvalues are all of the form $\lambda = \pm \frac{1}{\sinh u}$.
- (4) The eigenvalues of S are $\pm \lambda_k$, $2 \leq k \leq n$, where the λ_k are the eigenvalues of a tensor S' that measures the twist of the real foliation. See [F3] and Section 3.

Let ∇ be the connection associated with the metric $H_{\mathbb{C}}(\tau)$.

Proposition 2.8.

- (1) If $Z \in T^{1,0}M$, then $\nabla_Z \bar{\xi} = SZ$ and $\nabla_Z \xi = Z$. If $X \in T^{\mathbb{C}}M$, then $\nabla_X \frac{\partial}{\partial u} = \frac{1}{2}(X + SX)$.
- (2) If $Z \in T^{1,0}M$, then $R(\xi, \bar{\xi})Z = R(Z, \bar{\xi})\xi = -S^2Z$.
- (3) $2\text{Ric}(\xi, \bar{\xi}) = -\text{tr } S^2 = -\|S\|^2 = -2\|\bar{\partial}\xi\|^2$.

Proof.

- (1) The first statement is true by the definitions of S and ∇ . For the second, let W be a holomorphic vector field. Applying Z to $H_{\mathbb{C}}(\tau)(\xi, \bar{W}) = \bar{W}\tau$ we obtain

$$H_{\mathbb{C}}(\tau)(\nabla_Z \xi, \bar{W}) = Z\bar{W}\tau = \partial\bar{\partial}\tau(Z, \bar{W}) = H_{\mathbb{C}}(\tau)(Z, \bar{W}).$$

The third statement follows from the first two by writing $X = X^{1,0} + X^{0,1}$ and $\frac{\partial}{\partial u} = \frac{1}{2}(\xi + \bar{\xi})$. The special case $\nabla_{\xi}\xi = \xi$ appears in [S2], [B1], and [W1].

- (2) The first equality is a standard symmetry of the curvature tensor for a Kähler metric. For the other, let Z be a holomorphic vector field and use part (1) to obtain

$$\begin{aligned} R(Z, \bar{\xi})\xi &= \nabla_Z \nabla_{\bar{\xi}}\xi - \nabla_{\bar{\xi}} \nabla_Z \xi - \nabla_{[Z, \bar{\xi}]} \xi \\ &= 0 - 0 - \nabla_S Z \xi = -S^2 Z. \end{aligned}$$

- (3) By definition, $\text{Ric}(\xi, \bar{\xi}) = \text{tr}(R(\xi, \bar{\xi}) : T^{1,0}M \rightarrow T^{1,0}M)$. Thus by (2), $\text{Ric}(\xi, \bar{\xi}) = -\text{tr}(S^2|_{T^{1,0}M}) = -\sum \lambda_i^2$, where $\pm\lambda_2, \dots, \pm\lambda_n$ are the eigenvalues of S . (See Proposition 2.6(5) and the discussion following it.) By $\text{tr} S^2$ we mean

$$\text{tr}(S^2 : T^{\mathbb{C}}M \rightarrow T^{\mathbb{C}}M) = \text{tr}(S^2|_{T^{\mathbb{C}}M}) = 2\sum \lambda_i^2,$$

which accounts for the factor of 2 in the first equality. The second equality follows from the symmetry of S (Proposition 2.6(3)). The final equality follows from $S = \bar{\partial}\xi + \partial\bar{\xi}$ and the fact that $\bar{\partial}\xi$ and $\partial\bar{\xi}$ are orthogonal in $\Lambda^1 M \otimes T^{\mathbb{C}}M$. For other proofs of $\text{Ric}(\xi, \bar{\xi}) = -\|\bar{\partial}\xi\|^2$ see [S2] and [W1]. For a generalization, see [PW2]. \square

3. The Reinhardt Condition. In this section we introduce our assumption that is the link between the complex Monge-Ampère equation on a complex manifold and the real Monge-Ampère equation on an affine manifold. We study the local consequences of this assumption, and prove a version of Stoll's theorem on strictly parabolic manifolds as an application.

Recall that $f : \mathbb{C}^n \rightarrow \mathbb{R}$ is a Reinhardt function if $f(z^1 e^{i\theta^1}, \dots, z^n e^{i\theta^n}) = f(z^1, \dots, z^n)$ for all $\theta^k \in \mathbb{R}$. If $\zeta^\alpha = \log z^\alpha$, then f depends only on the real part of the coordinates $(\zeta^1, \dots, \zeta^n)$. This property is the main point, and so we make the following definition.

Definition 3.1. A function $f : M \rightarrow \mathbb{R}$ is *locally Reinhardt* if every point of M has a neighborhood with local coordinates $z = x + iy$ such that $f(z) = f(x)$, i.e., f is independent of $\text{Im } z$. Such coordinates are called *Reinhardt coordinates* for f .

Local Consequences. Let $u, \tau : M \rightarrow \mathbb{R}$ be as in Section 2. In addition, we assume that they are locally Reinhardt (if either is, both are). Examples 2.4 (1,2,4,5) are of this type, whereas Example 2.4(3) is not. (See Remark 3.3.) Let $z = x + iy$ be a Reinhardt coordinate system for u and τ on an open set $U \subset M$, and let u' and τ' denote the restrictions of u and τ to any slice of the coordinate system $U' = U \cap \{y = c\}$, $c \in \mathbb{R}$. Since

$$\frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{4} \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta} = \frac{1}{4} \frac{\partial^2 u'}{\partial x^\alpha \partial x^\beta},$$

the fact that u is a solution of

$$(3.1) \quad \text{rank } H_{\mathbb{C}}(u) = n - 1$$

implies that u' is a solution of

$$(3.2) \quad \text{rank } H(u') = n - 1.$$

Since u is *psh*, τ is *spsh*, and their level sets Σ are strictly pseudoconvex, it follows that u' is convex, τ' is strictly convex, and their level sets Σ' are strictly convex. This is relative, of course, to the Reinhardt coordinates. With an assumption on S , we will see (Theorem 3.5) that Reinhardt coordinates are unique in a sense, and then these properties will be global. These consequences of local Reinhardt-ness may be useful in giving this condition an invariant characterization. The author plans to address this in a future paper.

Next, we recall some facts about real Monge-Ampère foliations. (See [F3]). Let M' be a real affine manifold of dimension n , i.e., a manifold on which the changes of coordinates are affine transformations. There is a natural flat connection ∇' on M' defined by declaring all coordinate vector fields to be parallel. Let $u' : M' \rightarrow \mathbb{R}$ be C^5 . The real Hessian of u' is $H(u') = \nabla' du'$. We assume that u' is a solution of (3.2), and that $e^{u'}$ is strictly convex. Similar to the complex case, M' is foliated by straight lines such that u' is linear on each line. A vector $X \in TM'$ is tangent to the foliation if and only if $H(u')(X, \cdot) = 0$. Each level set Σ' of u' is strictly convex, the leaves of the foliation and the level sets are transverse, and $H(u')$ is positive definite on $T\Sigma'$. There is a unique vector field $\frac{\partial}{\partial u'}$ tangent to the foliation that satisfies $du'(\frac{\partial}{\partial u'}) = 1$. As in Definition 2.5, we define the tensor $S' = \nabla' \frac{\partial}{\partial u'}$, and think of S' as a vector bundle map $TM' \rightarrow TM'$. Clearly $S' \equiv 0$ if and only if $\frac{\partial}{\partial u'}$ is parallel. We have the following analogue of Proposition 2.6 [F3, Lemma 7, Corollary 8].

Proposition 3.2 (Properties of S' and $H(u')$).

- (1) $S' \frac{\partial}{\partial u'} = \nabla'_{\partial/\partial u'} \frac{\partial}{\partial u'} = 0$ and $\nabla_{\partial/\partial u'} du' = 0$.
- (2) S' takes its values in $T\Sigma'$, so we can view S' as a map $T\Sigma' \rightarrow T\Sigma'$.
- (3) S' is symmetric with respect to $H(u')$, and thus has $n-1$ eigenvalues.
- (4) $\nabla'_{\partial/\partial u'} S' = -S'^2$.
- (5) $(\nabla'_{\partial/\partial u'} H(u'))(X, Y) = -H(u')(S'X, Y)$.
- (6) $(\nabla'_X S')(Y) = (\nabla'_Y S')(X)$.
- (7) $(\nabla'_X H(u'))(Y, Z)$ is symmetric in X, Y, Z .
- (8) The eigenspaces of S' are ∇' -parallel along the leaves of the foliation.
- (9) The eigenvalues λ of S' satisfy $\frac{\partial \lambda}{\partial u'} = -\lambda^2$. Thus $\lambda = \frac{a}{au' + b}$, where a and b are parameters that depend continuously on the leaves. If a leaf crosses the level set $\Sigma'_0 = \{u' = 0\}$, then a and b can be scaled so that $b = 1$.
- (10) Let X be an eigenvector of S' that is ∇' -parallel along a leaf ℓ' of the foliation. If $S'X = \frac{a}{au' + b}X$ (a and b as in Part (9)), then $H(u')(X, X) = \frac{c}{au' + b}$ along ℓ' , where the magnitude of $|c|$ can be chosen by scaling X . (Note: In the applications of this, c will usually be determined by making X an $H(u')$ -unit vector all along ℓ' in the case $a = 0$, or at least at the point where ℓ' crosses $\{u' = 0\}$ when $a \neq 0$.)

Returning to the complex case, let $u, \tau, u', \tau', z = x + iy, U$, and U' be as in the beginning of this subsection. Decompose the real tangent bundle of U as $TU = T^xU \oplus T^yU$, where $T^xU = \text{span}\{\partial/\partial x^\alpha\}$ and $T^yU = \text{span}\{\partial/\partial y^\alpha\}$. Note that J interchanges T^xU and T^yU . For $X \in TU$, write $X = X_x + X_y$, where $X_x \in T^xU$ and $X_y \in T^yU$. On U we obtain a flat connection ∇' defined by declaring $\partial/\partial x^\alpha$ and $\partial/\partial y^\alpha$ to be parallel. On each slice U' we can define $\frac{\partial}{\partial u'} \in T^xU$ and $S' : T^xU \rightarrow T^xU$. For $X \in T^xU$, it is clear that $H(u')(X, \cdot) = 0$ if and only if $H_{\mathbb{C}}(u)(X, \cdot) = 0$. If ℓ is a leaf of the complex foliation, then $\ell \cap U'$ is a leaf of the real foliation on U' . Since $\frac{\partial}{\partial u}$ is the unique real vector field tangent to ℓ such that $du(\frac{\partial}{\partial u}) = 1$ and $du(J\frac{\partial}{\partial u}) = 0$ (see the discussion preceding Definition 2.5), we have $\frac{\partial}{\partial u} = \frac{\partial}{\partial u'}$. Then

$$\frac{\partial}{\partial u} \in T^xU, J\frac{\partial}{\partial u} \in T^yU,$$

$$S' : T\Sigma' \rightarrow T\Sigma' = T\Sigma \cap T^xU,$$

$$\nabla'_{X_y} \frac{\partial}{\partial u} = 0, \text{ and } \nabla'_{X_y} S' = 0.$$

Extend S' to all of $H\Sigma$ by setting $S'X = \nabla'_X \frac{\partial}{\partial u}$, i.e., $S' = 0$ on $H\Sigma \cap T^y U$. Since ∇' is the connection for the flat Kähler metric $\sum dz^\alpha \otimes d\bar{z}^\alpha$ on U , it follows that

$$\begin{aligned} SX &= \partial\bar{\xi}(X^{1,0}) + \bar{\partial}\xi(X^{0,1}) = \nabla'_{X^{1,0}}\bar{\xi} + \nabla'_{X^{0,1}}\bar{\xi} \\ &= \nabla'_X \frac{\partial}{\partial u} + J\nabla'_{JX} \frac{\partial}{\partial u} \end{aligned}$$

for $X \in H\Sigma$. Thus, $SX_x = S'X_x$ and $SX_y = JS'JX_y$.

It is now clear that the eigenvectors of S are in $T\Sigma'$ and $JT\Sigma'$, and the eigenvalues of $S : T\Sigma' \rightarrow T\Sigma'$ are the same as those of S' . The eigenvalues of S' satisfy $\frac{\partial\lambda}{\partial u'} = -\lambda^2$ by Proposition 3.2(9). Thus half of the eigenvalues of S satisfy $\frac{\partial\lambda}{\partial u} = -\lambda^2$ and the other half satisfy $\frac{\partial\lambda}{\partial u} = \lambda^2$ (see Proposition 2.6(5)).

Remark 3.3. As a consequence, we see that Example 2.4(3) is not locally Reinhardt, since $\lambda = \frac{1}{\sinh u}$ satisfies neither of these equations. (See Remark 2.7.)

Definition 3.4. The *primary eigenvalues* are those non-zero eigenvalues that satisfy $\frac{\partial\lambda}{\partial u} = -\lambda^2$. The corresponding eigenvectors are called *primary eigenvectors*.

The decomposition $TU = T^x U \oplus T^y U$ may seem a bit unnatural at first, however consider the following. Suppose that the eigenvalues of $S : H\Sigma \rightarrow H\Sigma$ in U are all non-zero. Then, *without reference to specific Reinhardt coordinates*, we see that $T^x U$ is the span of $\frac{\partial}{\partial u}$ and the primary eigenvectors, and $T^y U$ is the span of $J\frac{\partial}{\partial u}$ and the remaining eigenvectors. More generally, we have the following results.

Theorem 3.5. Suppose that no eigenvalue of $S : H\Sigma \rightarrow H\Sigma$ vanishes on an open set.

- (1) If z and w are both Reinhardt coordinates on some connected open set, then z and w are related by an affine change of coordinates $z^\alpha = \sum a^\alpha_\beta w^\beta + c^\alpha$, where (a^α_β) is a constant, real matrix, and each c^α is a complex constant.
- (2) ∇' is well-defined on M making M an affine manifold.
- (3) Each Reinhardt coordinate system is an affine coordinate system.
- (4) The decomposition $TM = T^x M \oplus T^y M$ is well-defined.
- (5) Let \mathcal{C} be an integral curve of $\frac{\partial}{\partial u}$. Then there is a neighborhood of \mathcal{C} that has a Reinhardt coordinate system (z^1, \dots, z^n) such that $u = \operatorname{Re} z^1$ on \mathcal{C} .

Proof. Let U be the common domain of z and w . It suffices to show that the matrix $(\partial z^\alpha / \partial w^\beta)$ is real on U . In proving this, arbitrary real linear changes of variables are permitted.

Let ℓ be a component of the intersection of U with a leaf of the complex foliation. Assume the eigenvalues are non-zero on ℓ (this holds for almost every leaf). Let $z = x + iy$ and $w = \nu + i\eta$, where $x, y, \nu, \eta \in \mathbb{R}^n$. For $p \in \ell$, the decomposition $T_p U = T_p^x U \oplus T_p^y U$ is well-defined. It follows that $u(x)$ and $u(\nu)$ induce the same map $S' : T_p^x \Sigma \rightarrow T_p^x \Sigma$.

Recall our convention for indices (Remark 2.1). In the real case, it is possible to find a linear change of coordinates such that on a fixed leaf x^2, \dots, x^n are constant, $\partial u / \partial x^1 = 1$ (i.e., $\partial / \partial x^1 = \partial / \partial u$), and $\partial / \partial x^2, \dots, \partial / \partial x^n$ form an $H(u)$ -orthogonal basis for $T_p \Sigma'$ of eigenvectors of S' for each p on the leaf. These claims follow from Proposition 3.2 (1–3, 8). Applying the same real linear change of coordinates to the coordinate system z yields another Reinhardt coordinate system (call it z again) such that on ℓ , we have that z^2, \dots, z^n are constant, $\partial u / \partial z^1 = \frac{1}{2} \partial u / \partial x^1 = \frac{1}{2}$ (i.e., $\partial / \partial z^1 = \frac{1}{2} \xi$), and $\partial / \partial z^2, \dots, \partial / \partial z^n$ form an $H_{\mathbb{C}}(u)$ -orthogonal basis for $T_p^{1,0} \Sigma$ of eigenvectors of S for each $p \in \ell$ (in particular, $\partial u / \partial z^k = \frac{1}{2} \partial u / \partial x^k = 0$ on ℓ). Assume the same has been done for the w coordinate system. By one final real linear change of coordinates, it can be assumed that $\partial / \partial x^k$ and $\partial / \partial \nu^k$ share the same eigenvalue λ_k . Adding a constant to u has no affect on (3.1), thus we may assume that ℓ meets the level set $\Sigma_0 = \{u = 0\}$. By Proposition 3.2(9), the primary eigenvalues have the form $\lambda_k = a_k / (a_k u + 1)$.

From

$$\frac{\partial}{\partial w^\beta} = \sum_{\alpha} \frac{\partial z^\alpha}{\partial w^\beta} \frac{\partial}{\partial z^\alpha},$$

it follows that $0 = \partial u / \partial w^k = \partial z^1 / \partial w^k \cdot \frac{1}{2}$ and $\frac{1}{2} = \partial u / \partial w^1 = \partial z^1 / \partial w^1 \cdot \frac{1}{2}$ on ℓ . Hence $\partial z^1 / \partial w^k = 0$ and $\partial z^1 / \partial w^1 = 1$ on ℓ .

In the leaf direction,

$$\begin{aligned} \frac{\partial^2 u}{\partial w^1 \partial \bar{w}^1} &= H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial w^1}, \frac{\partial}{\partial \bar{w}^1} \right) \\ &= \sum_{\alpha, \beta} \frac{\partial z^\alpha}{\partial w^1} \frac{\partial \bar{z}^\beta}{\partial \bar{w}^1} H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right). \end{aligned}$$

On ℓ it follows that

$$0 = H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial w^1}, \frac{\partial}{\partial \bar{w}^1} \right) = \sum_j \left| \frac{\partial z_j}{\partial w^1} \right|^2 H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \right).$$

Then $\partial z^j / \partial w^1 = 0$ on ℓ since $H_{\mathbb{C}}(u) (\partial / \partial z^j, \partial / \partial \bar{z}^j) > 0$ on ℓ .

On all of U we have

$$\begin{aligned}\frac{\partial^2 u}{\partial w^\gamma \partial \bar{w}^\gamma} &= H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial w^\gamma}, \frac{\partial}{\partial \bar{w}^\gamma} \right) \\ &= \sum_{\alpha, \beta} \frac{\partial z^\alpha}{\partial w^\gamma} \frac{\partial \bar{z}^\beta}{\partial \bar{w}^\gamma} H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\beta} \right) \\ &= \sum_{\alpha, \beta} \frac{\partial z^\alpha}{\partial w^\gamma} \frac{\partial \bar{z}^\beta}{\partial \bar{w}^\gamma} \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta}.\end{aligned}$$

Since u is real and independent of y and η , it follows that each $\partial u / \partial z^\alpha$, $\partial u / \partial w^\beta$ is real, and that $\partial^2 u / \partial z^\alpha \partial \bar{z}^\beta = \partial^2 u / \partial z^\alpha \partial z^\beta$ and $\partial^2 u / \partial w^\alpha \partial \bar{w}^\beta = \partial^2 u / \partial w^\alpha \partial w^\beta$ are real. Thus

$$\frac{\partial^2 u}{\partial w^\gamma \partial \bar{w}^\gamma} = \frac{\partial^2 u}{\partial (w^\gamma)^2} = \sum_{\alpha} \frac{\partial z^\alpha}{\partial (w^\gamma)^2} \frac{\partial u}{\partial z^\alpha} + \sum_{\alpha, \beta} \frac{\partial z^\alpha}{\partial w^\gamma} \frac{\partial \bar{z}^\beta}{\partial \bar{w}^\gamma} \frac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta}.$$

On ℓ in the directions tangent to the level sets we have

$$\begin{aligned}(3.3) \quad H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial w^k}, \frac{\partial}{\partial \bar{w}^k} \right) &= \sum_j \left| \frac{\partial z^j}{\partial w^k} \right|^2 H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \right) \\ &= \frac{1}{2} \frac{\partial^2 z^1}{\partial (w^k)^2} + \sum_j \left(\frac{\partial z^j}{\partial w^k} \right)^2 H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \right).\end{aligned}$$

Since $\partial / \partial x^k$ and $\partial / \partial \nu^k$ share the same eigenvalue λ_k on ℓ , it follows that if $\lambda_j \neq \lambda_k$, then

$$\begin{aligned}0 &= H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial w^k}, \frac{\partial}{\partial \bar{z}^j} \right) = \sum_i \frac{\partial z^i}{\partial w^k} H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right) \\ &= \frac{\partial z^j}{\partial w^k} H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \right),\end{aligned}$$

and so $\partial z^j / \partial w^k = 0$. Thus the sums in (3.3) are taken over only those j for which $\lambda_j = \lambda_k$. Assume $\partial / \partial z^2, \dots, \partial / \partial z^n$ and $\partial / \partial w^2, \dots, \partial / \partial w^n$ form $H_{\mathbb{C}}(u)$ -unitary bases of $T^{1,0}\Sigma_0$. Then

$$H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial w^k}, \frac{\partial}{\partial \bar{w}^k} \right) = H_{\mathbb{C}}(u) \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j} \right) = \frac{1}{\alpha_k u + 1}$$

when $\lambda_j = \lambda_k$ by Proposition 3.2(10). After multiplying by $a_k u + 1$, (3.3) becomes

$$1 = \sum_j \left| \frac{\partial z^j}{\partial w^k} \right|^2 = \frac{1}{2}(a_k u + 1) \frac{\partial^2 z^1}{\partial (w^k)^2} + \sum_j \left(\frac{\partial z^j}{\partial w^k} \right)^2.$$

Note that $u = x^1 = \nu^1$ on ℓ . Since $\partial z^j / \partial w^k$ and $\partial^2 z^1 / \partial (w^k)^2$ are holomorphic on ℓ and $a_k \neq 0$, it follows that $\partial^2 z^1 / \partial (w^k)^2 = 0$. Simple algebra then shows that the $\partial z^j / \partial w^k$ must all be real on ℓ .

For the general Reinhardt coordinates z and w , the matrix $(\partial z^\alpha / \partial w^\beta)$ is real on an open, dense subset of U . By continuity, it is real on all of U . This proves (1).

Statements (2–4) are direct consequences of part (1).

To prove (5), Let U be a simply connected neighborhood of \mathcal{C} . Let $p \in \mathcal{C}$. By part (1) we can find Reinhardt coordinates z^1, \dots, z^n on some neighborhood $V \subset U$ of p such that $u = \operatorname{Re} z^1$ on $\mathcal{C} \cap V$. The affine functions z^1, \dots, z^n have unique affine extensions to U . Shrinking U if necessary, these functions are the desired coordinates. \square

The condition on the eigenvalues is needed, as the following example shows. Let $u = \operatorname{Re} z^2 + (\operatorname{Re} z^1)^2 = x^2 + (x^1)^2$ on \mathbb{C}^2 . The leaves are the complex lines parallel to the z^2 axis, and $S \equiv 0$. If $z^1 = iw^1$ and $z^2 = w^2 + (w^1)^2$, then $x^1 = -\eta^1$, $x^2 = \nu^2 + (\nu^1)^2 - (\eta^1)^2$. Then $u = \nu^2 + (\nu^1)^2$, and so (w^1, w^2) is also a Reinhardt coordinate system.

Application to Strictly Parabolic Manifolds. We close this section by applying the Reinhardt condition to a special case of Stoll's theorem on strictly parabolic manifolds. Please recall the definitions, background, and notation from Section 1.

Proposition 3.6. *Let \widetilde{M} with $\tau : M \rightarrow [0, R^2)$, $0 < R \leq \infty$, be a strictly parabolic manifold in which τ is C^5 on \widetilde{M} and locally Reinhardt on $M = \widetilde{M} - \tau^{-1}(0)$. Then there is a biholomorphism $F : B_R^n \rightarrow \widetilde{M}$ such that $\tau \circ F = \tau_0$.*

Proof. All of the following steps except (5), which uses the Reinhardt condition, have appeared in [S2], [B1], and [W1]:

- (1) $\tau^{-1}(0)$ consists of a single point \mathcal{O} .
- (2) Each leaf is a punctured disk, with \mathcal{O} as the puncture.
- (3) Letting B_R^n be the R -ball in $T_{\mathcal{O}}\widetilde{M}$, the map $F = \operatorname{Exp}|_{B_R^n} : B_R^n \rightarrow \widetilde{M}$ is a diffeomorphism and holomorphic on the leaves.
- (4) F is a biholomorphism if and only if ξ is holomorphic.
- (5) ξ is holomorphic.

To prove (5), we show that $S = \bar{\partial}\xi + \partial\bar{\xi} \equiv 0$. Let ℓ be a leaf of the foliation. By Proposition 3.2(9), the eigenvalues of S on ℓ are $\lambda_i = \pm a_i/(a_i u + b_i)$. Let $r = \sqrt{\tau} = e^{u/2}$ be the geodesic distance to \mathcal{O} on \widetilde{M} . By Proposition 2.8(3) we have

$$\text{Ric}(\xi, \bar{\xi}) = -\frac{1}{2} \text{tr} S^2 = -\sum_i \frac{a_i^2}{(2a_i \log r + b_i)^2}.$$

This function must be C^1 on the disk $\ell \cup \mathcal{O}$, since τ is C^5 . This easily implies that each a_i must vanish. Thus $S \equiv 0$ and ξ is holomorphic. \square

4. The Embedding Theorem. This section contains the main results of the paper. We introduce our global hypotheses, which, taken together with the local results of Sections 2 and 3, yield a curvature estimate. This is combined with a result due to Demailly to prove the main embedding theorem.

The following definition generalizes Stoll's notion of a (strictly) parabolic manifold.

Definitions 4.1.

- (1) Let \widetilde{M} be a connected, complex manifold of dimension n , and let $\tau : \widetilde{M} \rightarrow [0, \infty)$ be a *sps* exhaustion. The pair (\widetilde{M}, τ) is said to be *strictly parabolic at infinity* if there is a compact subset $K \subset \widetilde{M}$ such that the function $u = \log \tau$ satisfies

$$(4.1) \quad (\partial\bar{\partial}u)^n = 0$$

on $M = \widetilde{M} - K$. If τ is C^k , we will say that (\widetilde{M}, τ) is C^k .

- (2) If τ is locally Reinhardt on M , then (\widetilde{M}, τ) is said to be of *Reinhardt type*.

The idea to consider manifolds that are strictly parabolic at infinity is due to Burns [B1-2], [F2]. The Reinhardt condition is a severe restriction. However, as we have seen, many specific examples do enjoy this property, namely Examples 2.4 (1, 2, 4, 5), the examples in [B2], and the tube domains studied by Patrizio [P3].

Remarks 4.2.

- (1) Since τ is proper, the set K can be chosen, for convenience, so that $\partial M = \partial K$ is a level set of τ . The strictly parabolic case is when $\partial M = \{\tau = 0\} = \{u = -\infty\}$.
- (2) When $\partial M = \{u = c\}$ for some $c > -\infty$, the number c can be set arbitrarily, since adding a constant to u does not affect anything in Definition 4.1.

- (3) Examples 2.4(2,3) do not fit Definition 4.1(1) exactly since τ is not differentiable on $K = \widetilde{M} - M$. The following easily-proved lemma allows τ to be smoothed at the expense of making K larger. Example 2.4(5) is locally Reinhardt on M , but is not strictly parabolic at infinity as $\widetilde{M} - M$ is not compact.

Lemma 4.3. *Let \widetilde{M} be Stein and suppose that $\varphi : \widetilde{M} \rightarrow [0, \infty)$ is a continuous exhaustion that is spsh on $\{\varphi \geq a\}$ for some $a > 0$. Then there exists a spsh exhaustion $\tilde{\varphi} : \widetilde{M} \rightarrow \mathbb{R}$ such that $\varphi = \tilde{\varphi}$ on $\{\varphi \geq a\}$. If φ is C^k , then $\tilde{\varphi}$ can be chosen to be C^k .*

The main results of the paper are the following two theorems, the second of which is a special case of the conjecture due to Burns (see Section 1).

Theorem 4.4. *Let (\widetilde{M}, τ) be strictly parabolic at infinity and C^4 . If there is a continuous function $\psi : \widetilde{M} \rightarrow \mathbb{R}$ and $\alpha, \beta \geq 0$ such that $\psi \leq \alpha(u - \log u) + \beta$ holds outside some compact set and such that the curvature estimate*

$$\text{Ric}_\tau + dd^c \psi > 0$$

holds on \widetilde{M} , then \widetilde{M} is biholomorphic to an algebraic submanifold of \mathbb{C}^{2n+1} . Here Ric_τ denotes the Ricci form for the τ -metric.

Theorem 4.5. *Let (\widetilde{M}, τ) be strictly parabolic at infinity, C^6 , and of Reinhardt type. In addition, assume on each component of $M = \widetilde{M} - K$ that the tensor S has (maximal) rank $n - 1$, or that the foliation is holomorphic ($S \equiv 0$). Then there exists a C^6 function $\psi_0 : \widetilde{M} \rightarrow \mathbb{R}$ with $\psi_0 = u - \log u$ on $\{u \geq 2\}$, and a sufficiently large constant N such that*

$$(4.2) \quad \text{Ric}_\tau + Ndd^c \psi_0 > 0$$

on \widetilde{M} . Taking $\psi = N\psi_0$ in Theorem 4.4, it follows that \widetilde{M} embeds algebraically in \mathbb{C}^{2n+1} .

The proof of Theorem 4.4 involves an adjustment of the geometry to accommodate the hypotheses of a result of Demailly (Theorem 4.6). The proof of the curvature estimate in Theorem 4.5 is a lengthy computation. The Reinhardt condition allows the use of the results from the real case (Section 3) to compute the Ricci tensor explicitly. If τ is C^k , then the tensor S is C^{k-3} . In the course of the proof three derivatives of S are taken, and so we need τ to be C^6 . We conclude this section with Example 4.6, showing that the curvature estimate can fail if the condition on S is dropped.

Theorem 4.6 (Demailly [D]). *Let \widetilde{M} be a connected, complex manifold of dimension n . Then \widetilde{M} is biholomorphic to an algebraic submanifold of \mathbb{C}^N if and only if there exists a C^4 spsh exhaustion $\varphi : \widetilde{M} \rightarrow [0, \infty)$ that satisfies the following conditions:*

- (1) $\int_{\widetilde{M}} (dd^c \varphi)^n < \infty$,
 - (2) $\{d\varphi = 0\}$ is finite, and
 - (3) There is a continuous $\psi : \widetilde{M} \rightarrow \mathbb{R}$ with $\psi < \alpha\varphi + \beta$ for some $\alpha, \beta \geq 0$ such that
- $$(4.3) \quad \text{Ric} + dd^c \psi > 0,$$

where Ric is the Ricci form for the metric with Kähler form $dd^c(e^\varphi)$. Furthermore, Ric may be replaced by the Ricci form of any metric with Kähler form Ω such that

$$(4.4) \quad e^{-(a\varphi+b)} dd^c(e^\varphi) \leq \Omega \leq e^{A\varphi+B} dd^c(e^\varphi)$$

for any $A, B, a, b \geq 0$.

The curvature estimate is, of course, the appropriate a-priori estimate for Hörmander's L^2 method for solving $\bar{\partial}$ -equations. (See, e.g., [Hö], [SY], [GW].)

Proof of Theorem 4.4. To use Theorem 4.6, we must define the function φ . It is tempting to let $\varphi = u$ on $\{u \geq 1\}$. This satisfies (1) of Theorem 4.6, but is not spsh. Instead, we use $\varphi = u - \log u$. By Remark 4.2(2), we assume that (4.1) holds on $\{u \geq 2\}$. Then φ is spsh on $\{u \geq 2\}$ and extends to a spsh function on all of \widetilde{M} by Lemma 4.3.

Since $\{u \leq 2\}$ is compact, $(dd^c \varphi)^n$ has a finite integral over it. On $\{u \geq 2\}$ we have

$$(dd^c \varphi)^n = \frac{n}{u^2} \left(1 - \frac{1}{u}\right)^{n-1} du \wedge d^c u \wedge (dd^c u)^{n-1}.$$

Stokes' Theorem and (4.1) imply that $d^c u \wedge (dd^c u)^{n-1}$ has the same integral C over every level set $\{u = c\}$ for $c \geq 2$. Applying Fubini's Theorem, we obtain

$$\int_{\{u \geq 2\}} (dd^c \varphi)^n = nC \int_2^\infty \frac{1}{u^2} \left(1 - \frac{1}{u}\right)^{n-1} du.$$

This is finite, and so hypothesis (1) of Theorem 4.6 is satisfied.

Hypothesis (2) is satisfied by making φ a Morse Function on $\{u \leq 2\}$.

For Hypothesis (3), we show that (4.4) holds for $\Omega = dd^c \tau$ and appropriate choices of the constants, implying that (4.2) can be substituted for (4.3). On $\{u \geq 2\}$ we have

$$dd^c(e^\varphi) = dd^c\left(\frac{\tau}{u}\right) = \frac{1}{u^2}((u-1)^2 + 1) du \wedge d^c u + \frac{1}{u}(u-1)dd^c u$$

and

$$dd^c \tau = \tau(du \wedge d^c u + dd^c u).$$

Comparing coefficients, one sees that

$$dd^c(e^\varphi) \leq dd^c \tau \leq e^\varphi dd^c(e^\varphi)$$

on $\{u \geq c\}$ for some sufficiently large $c \geq 2$. Since $\{u \leq c\}$ is compact and τ and e^φ are *spsh* on \widetilde{M} , constants $B, b \geq 0$ can be chosen so that

$$e^{-b} dd^c(e^\varphi) \leq dd^c \tau \leq e^{\varphi+B} dd^c(e^\varphi)$$

holds on all of \widetilde{M} , which is (4.4). By Theorem 4.6, we can identify \widetilde{M} with an algebraic submanifold X of C^N . Reducing the dimension of the ambient space to $2n+1$ is a simple exercise in algebraic geometry. See, e.g., [S]. \square

Next we prove the curvature estimate.

Proof of Theorem 4.5. Choose and fix a component M' of M . We first show that if c and N are sufficiently large, then (4.2) holds on $\{u \geq c\} \cap M'$.

If $z = (z^1, \dots, z^n)$ is a holomorphic coordinate system, then the Ricci form for the τ -metric is $\text{Ric}_\tau = -dd^c(\log \det(\tau_{\gamma\bar{\delta}}))$. (See, e.g., [He].) Then (4.2) is equivalent to

$$(4.5) \quad -H_{\mathbb{C}}(\log \det(\tau_{\gamma\bar{\delta}})) + NH_{\mathbb{C}}(\psi_0) > 0.$$

Now assume $z = x + iy$ is a Reinhardt coordinate system. If $f(z) = f(x + iy)$ is C^2 and independent of y , then $\partial^2 f / \partial z^\alpha \partial \bar{z}^\beta = \frac{1}{4} \partial^2 f / \partial x^\alpha \partial x^\beta$. For a fixed y , $H_{\mathbb{C}}(f)$ pulls back under $x \mapsto x + iy$ to $\frac{1}{4} H(f) = \frac{1}{4} \sum_{\alpha, \beta} f_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where the subscripts now (and for the rest of the proof) denote real derivatives. Then (4.5) pulls back to one-fourth of

$$(4.6) \quad R + NH(\psi_0) > 0,$$

where $R = -H(\log \det(\tau_{\gamma\bar{\delta}}))$.

Proving (4.6) is entirely a real variable matter. We will make much use of the results and notation of Section 3, especially Proposition 3.2. We will have no need for the connection of Section 2, and so the affine connection ∇' of Section 3 will be denoted by ∇ . To prove (4.6), we explicitly compute $R + NH(\psi_0)$. The result is (4.29).

Assume for the moment that $\text{rank}(S) = n - 1$ on M' . The case $S \equiv 0$ is considered in the paragraph containing (4.31). Let \mathcal{C} be an integral curve of $\frac{\partial}{\partial u}$. By Theorem 3.5(5) we may choose a Reinhardt coordinate system that covers \mathcal{C} . It will be shown later that such a coordinate system exists when $S \equiv 0$ as well, even though Theorem 3.5 cannot be used. Note that the computations leading to (4.29) are made relative to such a coordinate system, and so are valid when $S \equiv 0$. By a real linear change of coordinates (see the proof of Theorem 3.5(1)), we may assume along \mathcal{C} that $X_1 = \partial/\partial x^1 = \partial/\partial u$, and that $X_k = \partial/\partial x^k$, $k = 2, \dots, n$, are orthogonal primary eigenvectors of S . (Recall our convention for indices in Remark 2.1.) By Remark 4.2(2), we assume that \mathcal{C} meets $\{u = 0\}$ at some point p . Making X_2, \dots, X_n an $H(u)$ -orthonormal frame at p , we have by Proposition 3.2(9,10) that

$$(4.7) \quad H(u)(X_j, X_k) = u_{jk} = \frac{\delta_{jk}}{a_k u + 1}$$

along \mathcal{C} , where a_2, \dots, a_n are the eigenvalues of S at p . Each $a_k > 0$, since $\text{rank}(S) = n - 1$ and $H(u)$ is positive semi-definite. We also have that

$$H(u)(X_1, X_\alpha) = u_{1\alpha} = 0.$$

Define $R_{\alpha\beta}$ by $R = \sum_{\alpha,\beta} R_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where R is the tensor in (4.6). Noting that $(\tau_{\gamma\delta})$ is diagonal on \mathcal{C} , a routine computation yields

$$R_{\alpha\beta} = -(\log \det(\tau_{\gamma\delta}))_{\alpha\beta} = \sum_{\gamma,\delta} \frac{\tau_{\alpha\gamma\delta} \tau_{\beta\gamma\delta}}{\tau_{\gamma\gamma} \tau_{\delta\delta}} - \sum_{\gamma} \frac{\tau_{\alpha\beta\gamma\gamma}}{\tau_{\gamma\gamma}}$$

on \mathcal{C} . (All computations through (4.29) will be on \mathcal{C} .) Substituting $\tau = e^u$ and using the facts that $(u_{\alpha\beta})$ is diagonal, $u_1 = 1$, $u_k = 0$, and $u_{1\alpha} = u_{11\alpha} = u_{111\alpha} = 0$, one obtains

$$(4.8) \quad \begin{aligned} (1) \quad R_{11} &= \sum_k \frac{u_{1kk}^2}{u_{kk}^2} - \sum_k \frac{u_{11kk}}{u_{kk}}, \\ (2) \quad R_{1j} &= \sum_k \frac{u_{1kk} u_{jkk}}{u_{kk}^2} - \sum_k \frac{u_{1jkk}}{u_{kk}}, \\ (3) \quad R_{ii} &= -n u_{ii} + 2 u_{1ii} + 2 \frac{u_{1ii}^2}{u_{ii}} - u_{11ii} + \sum_{k,\ell} \frac{u_{ikk}^2}{u_{kk} u_{\ell\ell}} - \sum_k \frac{u_{iikk}}{u_{kk}}, \\ (4) \quad R_{ij} &= \sum_{k,\ell} \frac{u_{ikk} u_{jkk}}{u_{kk} u_{\ell\ell}} - \sum_k \frac{u_{ijkk}}{u_{kk}}, \quad \text{for } i \neq j. \end{aligned}$$

The terms u_{11ii} and $2u_{1ii}^2/u_{ii}$ in (4.8(3)) cancel as a result of (4.7). The above formulas are written collectively as

$$R_{\alpha\beta} = -nu_{\alpha\beta} + 2u_{1\alpha\beta} + \sum_{k,\ell} \frac{u_{\alpha k\ell} u_{\beta k\ell}}{u_{kk} u_{\ell\ell}} - \sum_k \frac{u_{\alpha\beta kk}}{u_{kk}}.$$

The rest of the proof involves computing expressions for $u_{\alpha\beta}$, $u_{\alpha\beta\gamma}$, $u_{\alpha\beta\gamma\delta}$, and $R_{\alpha\beta}$. Plugging $u_{kk} = \frac{1}{a_k u + 1}$ and $\lambda_k = \frac{a_k}{a_k u + 1}$ into (4.8(1)), the leaf curvature term becomes

$$(4.9) \quad R_{11} = -\sum_k \left(\frac{a_k}{a_k u + 1} \right)^2 = -\sum_k \lambda_k^2 = -\text{tr } S^2,$$

as expected.

Let X, Y, Z be affine parallel vector fields. Note that

$$(4.10) \quad \left[\frac{\partial}{\partial u}, X \right] = -SX \text{ and } [X, SY] = (\nabla_X S) Y.$$

Also,

$$\begin{aligned} (\nabla_{\partial/\partial u} \nabla_X H(u))(Y, Z) &= (\nabla_X \nabla_{\partial/\partial u} H(u))(Y, Z) - (\nabla_{SX} H(u))(Y, Z) \\ &= X((\nabla_{\partial/\partial u} H(u))(Y, Z)) - (\nabla_Y H(u))(SX, Z) \\ &= -X(H(u)(SY, Z)) - (\nabla_Y H(u))(SX, Z), \end{aligned}$$

by Proposition 3.2(5), and so

$$(4.11) \quad \begin{aligned} (\nabla_{\partial/\partial u} \nabla_X H(u))(Y, Z) &= -(\nabla_X H(u))(SY, Z) \\ &\quad - (\nabla_Y H(u))(SX, Z) - H(u)((\nabla_X S) Y, Z). \end{aligned}$$

Note that this is a tensor in Y and Z . From Proposition 3.2(4) we have

$$(4.12) \quad \begin{aligned} \nabla_{\partial/\partial u} \nabla_X S &= -\nabla_{SX} S - \nabla_X S^2 \\ &= -\nabla_{SX} S - (\nabla_X S) S - S(\nabla_X S). \end{aligned}$$

Equations (4.10), (4.11), and (4.12) will be used frequently in the rest of the proof.

From $u_{ijk} = (\nabla_{X_i} H(u))(X_j, X_k)$ and (4.11) it follows that

$$(4.13) \quad \begin{aligned} \frac{\partial}{\partial u} u_{ijk} &= (\nabla_{\partial/\partial u} \nabla_{X_i} H(u))(X_j, X_k) \\ &= -(\lambda_i + \lambda_j) u_{ijk} - H(u)((\nabla_{X_i} S) X_j, X_k) \\ &= -(\lambda_i + \lambda_j + \lambda_k) u_{ijk} - f_{ijk}(u), \end{aligned}$$

where $f_{ijk}(u) = H(u)((\nabla_{X_i} S)X_j, X_k) - \lambda_k u_{ijk}$ along \mathcal{C} . Note that $f_{ijk}(u)$ is symmetric in all three indices by (4.13).

We next find an expression for $f_{ijk}(u)$. Consider $H(u)((\nabla_{X_i} S)X_j, X_k)$ along \mathcal{C} :

$$\begin{aligned} \frac{\partial}{\partial u} (H(u)((\nabla_{X_i} S)X_j, X_k)) &= (\nabla_{\partial/\partial u} H(u))((\nabla_{X_i} S)X_j, X_k) + H(u)((\nabla_{\partial/\partial u} \nabla_{X_i} S)X_j, X_k) \\ &= -H(u)((\nabla_{X_i} S)X_j, SX_k) \\ &\quad - H(u)((\nabla_{SX_i} S) + (\nabla_{X_i} S)S + S(\nabla_{X_i} S))X_j, X_k) \\ &= -(\lambda_i + \lambda_j + 2\lambda_k) H(u)((\nabla_{X_i} S)X_j, X_k), \end{aligned}$$

by (4.12). Using $\lambda_i = \frac{a_i}{a_i u + 1}$, it follows easily that

$$(4.14) \quad H(u)((\nabla_{X_i} S)X_j, X_k) = \frac{A_{ijk}}{(a_i u + 1)(a_j u + 1)(a_k u + 1)^2},$$

where A_{ijk} is the value of $H(u)((\nabla_{X_i} S)X_j, X_k)$ on $\{u = 0\}$. Note that $A_{ijk} = A_{jik}$ by Proposition 3.2(6). We then have that

$$\begin{aligned} f'_{ijk}(u) &= -(\lambda_i + \lambda_j + 2\lambda_k) H(u)((\nabla_{X_i} S)X_j, X_k) + \lambda_k^2 u_{ijk} - \lambda_k \frac{\partial}{\partial u} u_{ijk} \\ &= -(\lambda_i + \lambda_j + \lambda_k) f_{ijk}(u), \end{aligned}$$

and so

$$(4.15) \quad f_{ijk}(u) = \frac{\tilde{A}_{ijk}}{(a_i u + 1)(a_j u + 1)(a_k u + 1)}.$$

Note that \tilde{A}_{ijk} is symmetric in all three indices since $f_{ijk}(u)$ is.

Plugging into (4.13) we obtain

$$\frac{\partial}{\partial u} u_{ijk} = -(\lambda_i + \lambda_j + \lambda_k) u_{ijk} - \frac{\tilde{A}_{ijk}}{(a_i u + 1)(a_j u + 1)(a_k u + 1)}.$$

This can be integrated, after multiplication by $(a_i u + 1)(a_j u + 1)(a_k u + 1)$, yielding

$$(4.16) \quad u_{ijk} = \frac{\tilde{A}_{ijk} u + B_{ijk}}{(a_i u + 1)(a_j u + 1)(a_k u + 1)}.$$

Note that B_{ijk} is symmetric in all three indices.

We now have two expressions for $H(u)((\nabla_{X_i} S)X_j, X_k)$, namely (4.14) and another obtained by combining (4.13) with (4.16) and its $\frac{\partial}{\partial u}$ derivative. As a result,

$$(4.17) \quad A_{ijk} = \tilde{A}_{ijk} + a_k B_{ijk}.$$

We can now compute the “cross” tangential-leaf curvature terms. Plugging (4.16) and its $\frac{\partial}{\partial u}$ derivative (which is u_{1ijk}) into (4.8(2)), we obtain

$$R_{1j} = \sum_k \frac{\tilde{A}_{jkk} - a_j a_k u^2 \tilde{A}_{jkk} + a_j B_{jkk} (a_k u + 1) + a_k B_{jkk} (a_j u + 1)}{(a_j u + 1)^2 (a_k u + 1)^2},$$

which, using (4.17), can be rewritten as

$$(4.18) \quad R_{1j} = \sum_k \frac{A_{jkk} (a_j u + 1) + A_{kkj} (a_k u + 1) - \tilde{A}_{jkk} (a_j u + 1) (a_k u + 1)}{(a_j u + 1)^2 (a_k u + 1)^2}.$$

Next, we compute $u_{ijk\ell} = (\nabla_{X_i} \nabla_{X_j} H(u))(X_k, X_\ell)$. Using (4.10) and (4.11) we obtain

$$\begin{aligned} \frac{\partial}{\partial u} u_{ijk\ell} &= (\nabla_{\partial/\partial u} \nabla_{X_i} \nabla_{X_j} H(u))(X_k, X_\ell) \\ &= -\lambda_i u_{ijk\ell} + (\nabla_{X_i} \nabla_{\partial/\partial u} \nabla_{X_j} H(u))(X_k, X_\ell) \\ &= -\lambda_i u_{ijk\ell} + X_i((\nabla_{\partial/\partial u} \nabla_{X_j} H(u))(X_k, X_\ell)) \\ &= -\lambda_i u_{ijk\ell} - X_i((\nabla_{X_j} H(u))(SX_k, X_\ell)) \\ &\quad - X_i((\nabla_{X_k} H(u))(SX_j, X_\ell) + H(u)((\nabla_{X_j} S)X_k, X_\ell)) \\ &= -(\lambda_i + \lambda_j + \lambda_k) u_{ijk\ell} - (\nabla_{X_i} H(u))((\nabla_{X_j} S)X_k, X_\ell) \\ &\quad - (\nabla_{X_j} H(u))((\nabla_{X_k} S)X_i, X_\ell) - (\nabla_{X_k} H(u))((\nabla_{X_i} S)X_j, X_\ell) \\ &\quad - H(u)((\nabla_{X_i} \nabla_{X_j} S)X_k, X_\ell). \end{aligned}$$

Then

$$(4.19) \quad \frac{\partial}{\partial u} u_{ijk\ell} = -(\lambda_i + \lambda_j + \lambda_k + \lambda_\ell) u_{ijk\ell} - f_{ijk\ell}(u),$$

where

$$\begin{aligned} (4.20) \quad f_{ijk\ell}(u) &= (\nabla_{X_i} H(u))((\nabla_{X_j} S)X_k, X_\ell) + (\nabla_{X_j} H(u))((\nabla_{X_k} S)X_i, X_\ell) \\ &\quad + (\nabla_{X_k} H(u))((\nabla_{X_i} S)X_j, X_\ell) \\ &\quad + H(u)((\nabla_{X_i} \nabla_{X_j} S)X_k, X_\ell) - \lambda_\ell u_{ijk\ell}. \end{aligned}$$

Note that $f_{ijk\ell}(u)$ is symmetric in all four indices by (4.19).

To get an expression for $f'_{ijk\ell}(u)$ we compute the $\frac{\partial}{\partial u}$ derivatives of the first four terms on the right side of (4.20). (Note that S is C^3 since τ is C^6 . Thus $f_{ijk\ell}$ is differentiable.)

Let $g_{ijk\ell}(u) = (\nabla_{X_i} H(u)) ((\nabla_{X_j} S) X_k, X_\ell)$. Then

$$\begin{aligned} g'_{ijk\ell}(u) &= (\nabla_{\partial/\partial u} \nabla_{X_i} H(u)) ((\nabla_{X_j} S) X_k, X_\ell) \\ &\quad + (\nabla_{X_i} H(u)) ((\nabla_{\partial/\partial u} \nabla_{X_j} S) X_k, X_\ell). \end{aligned}$$

Using (4.11) with $X = X_i$, $Y = X_\ell$, $Z = (\nabla_{X_j} S) X_k$, and (4.12), one obtains

$$\begin{aligned} g'_{ijk\ell}(u) &= -(\lambda_i + \lambda_j + \lambda_k + \lambda_\ell) g_{ijk\ell}(u) - H(u) ((\nabla_{X_i} S) X_\ell, (\nabla_{X_j} S) X_k) \\ &\quad - (\nabla_{X_i} H(u)) (S(\nabla_{X_j} S) X_k, X_\ell). \end{aligned}$$

Let X, Y, Z be parallel fields. Then

$$\begin{aligned} &(\nabla_X H(u)) (SY, Z) - (\nabla_X H(u)) (Y, SZ) \\ &= X(H(u)(SY, Z)) - H(u)((\nabla_X S)Y, Z) \\ &\quad - X(H(u)(SY, Z)) + H(u)(Y, (\nabla_X S)Z), \end{aligned}$$

and so

$$\begin{aligned} (\nabla_X H(u)) (SY, Z) &= (\nabla_X H(u)) (Y, SZ) \\ &\quad + H(u)(Y, (\nabla_X S)Z) - H(u)((\nabla_X S)Y, Z), \end{aligned}$$

which is a tensor in X, Y, Z . Letting $X = X_i$, $Y = (\nabla_{X_j} S) X_k$, $Z = X_\ell$, it follows that

$$\begin{aligned} (4.21) \quad g'_{ijk\ell}(u) &= -(\lambda_i + \lambda_j + \lambda_k + 2\lambda_\ell) g_{ijk\ell}(u) \\ &\quad - 2H(u)((\nabla_{X_i} S) X_\ell, (\nabla_{X_j} S) X_k) \\ &\quad + H(u)((\nabla_{X_i} S) (\nabla_{X_j} S) X_k, X_\ell). \end{aligned}$$

Let $h(u) = H(u)((\nabla_{X_i} \nabla_{X_j} S) X_k, X_\ell)$. Then

$$\begin{aligned} h'(u) &= -\lambda_\ell h(u) + H(u)((\nabla_{\partial/\partial u} \nabla_{X_i} \nabla_{X_j} S) X_k, X_\ell) \\ &= -(\lambda_i + \lambda_\ell) h(u) + H(u)((\nabla_{X_i} \nabla_{\partial/\partial u} \nabla_{X_j} S) X_k, X_\ell). \end{aligned}$$

Using (4.12), this becomes

$$\begin{aligned} h'(u) &= -(\lambda_i + \lambda_\ell) h(u) \\ &\quad - H(u) ((\nabla_{X_i} \nabla_{S X_j} S + (\nabla_{X_i} \nabla_{X_j} S) S + (\nabla_{X_j} S)(\nabla_{X_i} S)) X_k, X_\ell) \\ &\quad - H(u) (((\nabla_{X_i} S)(\nabla_{X_j} S) + S(\nabla_{X_i} \nabla_{X_j} S)) X_k, X_\ell), \end{aligned}$$

which simplifies to

$$\begin{aligned} (4.22) \quad h'(u) &= -(\lambda_i + \lambda_j + \lambda_k + 2\lambda_\ell) h(u) \\ &\quad - H(u) ((\nabla_{X_i} S)(\nabla_{X_j} S) X_k, X_\ell) \\ &\quad - H(u) ((\nabla_{X_j} S)(\nabla_{X_k} S) X_i, X_\ell) \\ &\quad - H(u) ((\nabla_{X_k} S)(\nabla_{X_i} S) X_j, X_\ell). \end{aligned}$$

Cyclically permute i, j, k in (4.21). Adding the sum of the resulting equations to (4.22) and using (4.19) yields the derivative of (4.20):

$$\begin{aligned} (4.23) \quad f'_{ijk\ell}(u) &= -(\lambda_i + \lambda_j + \lambda_k + \lambda_\ell) f_{ijk\ell}(u) \\ &\quad - 2H(u) ((\nabla_{X_i} S) X_\ell, (\nabla_{X_j} S) X_k) \\ &\quad - 2H(u) ((\nabla_{X_j} S) X_\ell, (\nabla_{X_k} S) X_i) \\ &\quad - 2H(u) ((\nabla_{X_k} S) X_\ell, (\nabla_{X_i} S) X_j). \end{aligned}$$

Recalling (4.7), equation (4.14) implies that

$$((\nabla_{X_i} S) X_j)^{\tan} = \sum_r \frac{A_{ijr}}{(a_i u + 1)(a_j u + 1)(a_r u + 1)} X_r,$$

where X^{\tan} means the component of X that is tangent to the level set Σ of u with respect to the decomposition $TM = T\Sigma \oplus \langle \frac{\partial}{\partial u} \rangle$. Equation (4.23) then becomes

$$\begin{aligned} (4.24) \quad f'_{ijk\ell}(u) &= -(\lambda_i + \lambda_j + \lambda_k + \lambda_\ell) f_{ijk\ell}(u) \\ &\quad - \frac{2}{(a_i u + 1)(a_j u + 1)(a_k u + 1)(a_\ell u + 1)} \sum_r \frac{C_{ijk\ell r}}{(a_r u + 1)^3}, \end{aligned}$$

where $C_{ijklr} = A_{i\ell r}A_{jkr} + A_{j\ell r}A_{kir} + A_{k\ell r}A_{ijr}$. Note that C_{ijklr} is symmetric in i, j, k, ℓ .

Equations (4.24) and (4.19) can be rewritten as

$$(4.25) \quad \frac{\partial}{\partial u} \left(\frac{f_{ijkl}(u)}{u_{ii}u_{jj}u_{kk}u_{\ell\ell}} \right) = -2 \sum_r \frac{C_{ijklr}}{(a_r u + 1)^3}$$

and

$$(4.26) \quad \frac{\partial}{\partial u} \left(\frac{u_{ijkl}}{u_{ii}u_{jj}u_{kk}u_{\ell\ell}} \right) = -\frac{f_{ijkl}(u)}{u_{ii}u_{jj}u_{kk}u_{\ell\ell}}.$$

These yield

$$(4.27) \quad \frac{f_{ijkl}(u)}{u_{ii}u_{jj}u_{kk}u_{\ell\ell}} = -\sum_r C_{ijklr} \frac{2u + a_r u^2}{(a_r u + 1)^2} - D_{ijkl}$$

and

$$(4.28) \quad \frac{u_{ijkl}}{u_{ii}u_{jj}u_{kk}u_{\ell\ell}} = \sum_r \frac{C_{ijklr} u^2}{a_r u + 1} + D_{ijkl} u + E_{ijkl},$$

where $D_{ijkl} = -f_{ijkl}(0)$ and $E_{ijkl} = u_{ijkl}|_{u=0}$.

Now (4.16) and (4.28) can be used in (4.8(3,4)) to obtain expressions for the “tangential” curvature terms. Combining this with the “leaf” term (4.9) and the “cross” terms (4.18), and recalling that $\psi_0 = u - \log u$, we obtain

$$(4.29) \quad R + NH(\psi_0) = \sum_{\alpha, \beta} \tilde{R}_{\alpha\beta} dx^\alpha \otimes dx^\beta.$$

where

$$\begin{aligned} \tilde{R}_{11} &= \frac{N}{u^2} - \frac{1}{2} \operatorname{tr} S^2 = \frac{N}{u^2} - \sum_i \left(\frac{a_i}{a_i u + 1} \right)^2, \\ \tilde{R}_{1j} &= \sum_k \frac{A_{jkk}(a_j u + 1) + A_{kkj}(a_k u + 1) - \tilde{A}_{jkk}(a_j u + 1)(a_k u + 1)}{(a_j u + 1)^2 (a_k u + 1)^2}, \end{aligned}$$

and

$$\begin{aligned} \tilde{R}_{ij} = & \left(\left(N - n - \frac{N}{u} \right) \frac{1}{a_i u + 1} + \frac{2a_i}{(a_i u + 1)^2} \right) \delta_{ij} \\ & + \sum_k \frac{1}{(a_i u + 1)(a_j u + 1)(a_k u + 1)} \\ & \times \left(\sum_\ell \frac{(\tilde{A}_{ik\ell} u - B_{ik\ell})(\tilde{A}_{jk\ell} u - B_{jk\ell}) - C_{ijk\ell} u^2}{a_\ell u + 1} - D_{ijkk} u - E_{ijkk} \right). \end{aligned}$$

Note that the dominant tangential term is

$$(N - n) \sum_i \frac{1}{a_i u + 1} (dx^i)^2 = (N - n)H(u).$$

Let $N > n$ so that this is positive definite on the tangent spaces of the level sets.

Recall (see (4.7)) that a_2, \dots, a_n are the eigenvalues of S on the compact set $\{u = 0\} \cap M'$, and that they are positive. Thus we may choose $\alpha, a \in \mathbb{R}$, independent of \mathcal{C} , such that $0 < \alpha < \min\{1, a_k\}$ and $a > \max\{a_k\}$. The coordinate-dependent terms in (4.29) (A_{ijk}, \tilde{A}_{ijk} , etc.) are the values of various tensors at an orthonormal frame on $\{u = 0\} \cap M'$ (see (4.14), (4.16), (4.17), (4.20), (4.24), (4.27), and (4.28)). Since these tensors are independent of the coordinates (Theorem 3.5), and the bundle of orthonormal frames along $\{u = 0\}$ is compact, we may choose $A \in \mathbb{R}$, independent of \mathcal{C} , such that $A > \max\{|A_{ijk}|, |\tilde{A}_{ijk}|, |B_{ijk}|, |C_{ijk\ell}|, |D_{ijk\ell}|, |E_{ijk\ell}|\}$. If c is chosen sufficiently large, then $u \geq c$ implies that

$$|\tilde{R}_{1j}| \leq \frac{Aa^2}{\alpha^4 u^2} \equiv \frac{C_1}{u^2},$$

and

$$\left| \tilde{R}_{ij} - \frac{N - n}{a_i u + 1} \delta_{ij} \right| \leq \frac{1}{u^2} \left(\frac{N}{\alpha} + \frac{2a}{\alpha^2} + \frac{A^2 + 2A}{\alpha^4} \right) \equiv \frac{C_2}{u^2}.$$

(This second estimate uses $\alpha < 1$.) For the remaining terms we have,

$$\frac{N}{u^2} - \sum_i \left(\frac{a_i}{a_i u + 1} \right)^2 > \frac{N - n}{u^2},$$

and

$$(N - n) \sum_i \frac{1}{a_i u + 1} (dx^i)^2 > \frac{N - n}{au} \sum_i (dx^i)^2.$$

We thus have that

$$\begin{aligned}
 (4.30) \quad R + NH(\psi_0) &> \frac{N-n}{u^2} (dx^1)^2 - \frac{2C_1}{u^2} \sum_k dx^1 \cdot dx^k + \left(\frac{N-n}{au} - \frac{2C_2}{u^2} \right) \sum_k (dx^k)^2 \\
 &= \frac{N-n}{u^2} \left((dx^1)^2 - \frac{2C_1}{N-n} \sum_k dx^1 \cdot dx^k + \left(\frac{u}{a} - \frac{2C_2}{N-n} \right) \sum_k (dx^k)^2 \right),
 \end{aligned}$$

where $dx^1 \cdot dx^k = \frac{1}{2}(dx^1 \otimes dx^k + dx^k \otimes dx^1)$. Choosing c large enough so that $u \geq c$ implies $\frac{u}{a} - 2C_2/(N-n) \geq (C_1/(N-n))^2$, we see that the right hand side of (4.30) is positive, proving the curvature estimates (4.6) and (4.2) on $\{u \geq c\} \cap M'$ when $\text{rank}(S) = n-1$.

Now assume $S \equiv 0$ on M' . As before, we may assume that $\{u = 0\}$ is contained in M . Given $p \in \{u = 0\}$, let \mathcal{C}_p denote the intersection of $\{u \geq 0\}$ and the integral curve of $\frac{\partial}{\partial u}$ passing through p . Suppose that a Reinhardt coordinate chart can be chosen that contains \mathcal{C}_p . Tracing through the computation leading to (4.29) (it begins in the paragraph containing (4.7)), one sees that $a_k = A_{ijk} = \tilde{A}_{ijk} = C_{ijk\ell r} = D_{ijk\ell} = 0$ (see (4.14), (4.16), (4.17), (4.20), (4.24), (4.27), and (4.28)). Then (4.29) becomes

$$\begin{aligned}
 (4.31) \quad R + NH(\psi_0) &= \frac{N}{u^2} (dx^1)^2 + \left(N - n - \frac{N}{u} \right) \sum_i (dx^i)^2 \\
 &\quad + \sum_{i,j,k} \left(\sum_{\ell} B_{ik\ell} B_{jk\ell} - E_{ijkk} \right) dx^i \cdot dx^j.
 \end{aligned}$$

Since Theorem 3.5 does not apply, a global affine structure on M' may not be defined, and so the tensors involved in (4.31), namely $H(u)$ and its affine covariant derivatives, depend on the Reinhardt coordinates. Suppose, however, that finitely many Reinhardt coordinate charts U_1, \dots, U_m can be chosen to cover $\{u \geq 0\} \cap M'$ such that whenever $p \in U_i \cap \{u = 0\}$, then $\mathcal{C}_p \subset U_i$. It is clear from (4.31) that N can be chosen sufficiently large so (4.6) and (4.2) hold on $\{u \geq 2\}$.

To see that these coordinate charts can be chosen, recall that in the real case $S' \equiv 0$ if and only if $\frac{\partial}{\partial u'}$ is affine parallel (Section 3). The flow of $\frac{\partial}{\partial u'}$ is then an affine translation. Since the complex Reinhardt case can be written locally as $u(z) = u'(x)$ for some u' satisfying the real case (Example 2.4(4)) and the inclusion $x \mapsto x + iy$ identifies $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial u'}$, it follows that the flow of $\frac{\partial}{\partial u}$ is a

translation in Reinhardt coordinates when $S \equiv 0$. Let Φ_t denote the flow of $\frac{\partial}{\partial u}$ for $t \geq 0$. Then Φ_t is a local biholomorphism.

Let $p \in \{u = 0\}$, and let z denote local Reinhardt coordinates on a neighborhood of p . Then $z_t = z \circ \Phi_{-t}$ is a local holomorphic coordinate chart on a neighborhood of $\Phi_t(p)$. Moreover, we have

$$\frac{\partial u}{\partial z_t^k} = (\Phi_t)_* \left(\frac{\partial}{\partial z^k} \right) u = \frac{\partial}{\partial z^k} (u \circ \Phi_t) = \frac{\partial}{\partial u} (u + t) = \frac{\partial u}{\partial z^k}.$$

Thus $\text{Im } \partial u / \partial z_t^k = 0$, and so z_t is a Reinhardt coordinate system. If the coordinate charts z_t and z_s meet, then the coordinate change is given by translation in either coordinate system. It follows that the initial coordinates $z_0 = z$ can be extended to an open set U with the property that if $p \in U \cap \{u = 0\}$, then $\mathcal{C}_p \subset U$. Since $\{u = 0\} \cap M'$ is compact, it can be covered with finitely many coordinate neighborhoods of this type, and these neighborhoods cover $\{u \geq 0\} \cap M'$.

In both cases, $\text{rank}(S) = n - 1$ and $S \equiv 0$, we have found $c > 2$ and $N > 0$ such that (4.2) holds on $\{u \geq c\} \cap M'$. Since $\{u = 0\}$ is compact and each component of M contains a component of $\{u = 0\}$, M has finitely many components, and so a single N can be chosen to work for all of them. Since $\psi_0 = u - \log u$ is *sps*h on $\{u \geq 2\}$, ψ_0 can be extended to be a *sps*h function on all of \widetilde{M} by Lemma 4.3. Then, since $\text{Ric}_\tau + N dd^c \psi_0 > 0$ holds outside a compact set, we can make N sufficiently large so that this holds on all of \widetilde{M} .

This completes the proof of Theorem 4.5. \square

The following is an example in which the curvature estimate (4.2) fails on $\{u \geq c\}$ for every choice of c and N . Note that S drops rank on one of the leaves. Since any real solution can be complexified by the method in Example 2.4(4), it suffices to give a real example in which (4.6) fails.

Example 4.6. In \mathbb{R}^2 , let $\Sigma_0 = \{x^2 + y^2 = 1\}$ and $\Sigma_{-1} = \{4x^2 + 16y^2 = 1\}$. Let u be the unique solution of $\det H(u) \equiv 0$ on $\{4x^2 + 16y^2 \geq 1\}$ with $u = 0$ on Σ_0 and $u = -1$ on Σ_{-1} . This function can be constructed using Example 6(2) of [F3], where N is the vector field along Σ_0 defined as follows. For $p \in \Sigma_0$, let $q \in \Sigma_{-1}$ be the closer to p of the two points for which $T_q \Sigma_{-1}$ is parallel to $T_p \Sigma_0$. Set $N_p = p - q$. The leaves of the foliation are the rays $\{p + tN_p \mid t \geq -1\}$ for $p \in \Sigma_0$. This construction leads to the only solution, since the tangent spaces of the level sets of u must be parallel along the leaves of the foliation. Furthermore, the solution is real analytic. Along Σ_0 we have $\frac{\partial}{\partial u} = N$. If Σ_0 is parameterized by $\sigma(t) = (\cos t, \sin t)$, then the eigenvalue of S at $\sigma(t)$ is $a(t) = 1 - (3 \cos^2 t + 1)^{-3/2}$. We have $a(t) \geq 0$, but $a(\pm \frac{\pi}{2}) = 0$, and so S drops rank along the y -axis, which is a leaf. The parameters $A = A_{111}$, $\tilde{A} = \tilde{A}_{111}$, $B = B_{111}$, $C_{11111} = 3A^2$,

$D = D_{1111}$, and $E = E_{1111}$ can all be explicitly computed as functions of t . It turns out that $A, \bar{A} \in \mathcal{O}(\sqrt{a})$, $B \in \mathcal{O}(a)$, and that $D(\pm \frac{\pi}{2})$ and $E(\pm \frac{\pi}{2})$ are non-zero. It follows that there is no set $\{u \geq c\}$ on which $R + NH(\psi_0) > 0$ for any choice of N .

One might think that requiring S to have constant rank k , $0 < k < n-1$, might imply the curvature estimate. However under the assumption that some, but not all, of the eigenvalues vanish, and without additional information about the parameters A_{ijk}, \bar{A}_{ijk} , etc., the cross terms in (4.29) do not decay quickly enough for $R + NH(\psi_0) > 0$ to hold even on a fixed integral curve \mathcal{C} of $\frac{\partial}{\partial u}$.

REFERENCES

- [BB] E. BEDFORD & D. M. BURNS, *Holomorphic Mapping of Annuli in C^n and the Associated Extremal Function*, Annali Della Scuola Normale Superiore Di Pisa, Serie IV **6** (1979), 381–414.
- [BK] E. BEDFORD & M. KALKA, *Foliations and Complex Monge-Ampère Equations*, Communications on Pure and Applied Mathematics **30** (1977), 543–571.
- [B1] D. M. BURNS, *Curvatures of Monge-Ampère Foliations and Parabolic Manifolds*, Annals of Mathematics **115** (1982), 349–374.
- [B2] D. M. BURNS, *Les Equations de Monge-Ampère Homogeneous et des Exhaustions Speciales de Varietes Affines*, Sem. Goulaouic-Meyer-Schwartz 1983-84, No.4, Ecole Polytechnique Palaiseau (1984).
- [D] J. P. DEMAILLY, *Mesures de Monge-Ampère et Caracterisation Geometrique des Varietes Algebriques Affines*, Mém. Soc. Math. France (N.S.) **19** (1985).
- [F1] R. L. FOOTE, *Curvature Estimates for Monge-Ampère Foliations*, Ph.D. Thesis, Univ. of Michigan, 1983.
- [F2] R. L. FOOTE, *Stein Manifolds that Admit Monge-Ampère Foliations*, Proceedings of the Third International Conference on Complex Analysis and Applications, Varna, Bulgaria (1985), 220–227.
- [F3] R. L. FOOTE, *Differential Geometry of Real Monge-Ampère Foliations*, Mathematische Zeitschrift **194** (1987), 331–350.
- [Fr] M. FREEMAN, *The Levi Form and Local Complex Foliations*, Proc. Amer. Math. Soc. **57** (1976), 369–370.
- [GW] R. E. GREENE & H. WU, *A Theorem in Complex Geometric Function Theory*, in: *Value-Distribution Theory, Part A*, R.O. Kujala and A.L. Vitter, eds. (New York: Marcel Dekker, 1973), pp. 145–167.
- [He] S. HELGASON, *Differential Geometry Lie Groups and Symmetric Spaces*, New York: Academic Press, 1978.
- [Hö] L. HÖRMANDER, *An Introduction to Complex Analysis in Several Variables*, 2nd ed. Amsterdam: North Holland, 1973.
- [L1] L. LEMPERT, *La Métrique de Kobayshi et la Représentation des Domaines Sur la Boule*, Bull. Soc. Math France **109** (1981), 427–474.
- [L2] L. LEMPERT, *Solving the Degenerate Complex Monge-Ampère Equation with One Concentrated Singularity*, Math. Annalen **263** (1983), 515–532.
- [P1] G. PATRIZIO, *Parabolic Exhaustions for Strictly Convex Domains*, Manuscripta Math. **47** (1984), 271–309.

- [P2] G. PATRIZIO, *A Characterization of Complex Manifolds Biholomorphic to a Circular Domain*, Math. Zeit. **189** (1985), 343-363.
- [P3] G. PATRIZIO, *Real and Complex Monge-Ampère Equations and the Geometry of Strictly Convex Domains*, Publ. RIMS Kyoto Univ. **20** (1984), 867-875.
- [P4] G. PATRIZIO, *The Kobayashi Metric and the Homogeneous Complex Monge-Ampère Equation*, Proceedings of the Third International Conference on Complex Analysis and Applications, Varna, Bulgaria (1985), 515-523.
- [PW1] G. PATRIZIO & P.M. WONG, *Stability of the Monge-Ampère Foliation*, Math. Ann. **263** (1983), 13-29.
- [PW2] G. PATRIZIO & P.M. WONG, *Stein Manifolds with Compact Symmetric Centers*, (Preprint).
- [S] I. R. SHAFAREVICH, *Basic Algebraic Geometry*, New York: Springer-Verlag, 1977.
- [SY] Y.-T. SIU & S.-T. YAU, *Complete Kähler Manifolds with Nonpositive Curvature of Faster than Quadratic Decay*, Annals of Mathematics **105** (1977), 225-264.
- [S1] W. STOLL, *Value Distribution on Parabolic Spaces*, New York: Springer-Verlag, 1977.
- [S2] W. STOLL, *The characterization of strictly parabolic manifolds*, Annali Della Scuola Normale Superiore Di Pisa, Serie IV **7** (1980), 87-154.
- [S3] W. STOLL, *The characterization of strictly parabolic spaces*, Comp. Math. **144** (1981), 305-373.
- [W1] P. M. WONG, *Geometry of the complex homogeneous Monge-Ampère equation*, Inventiones Mathematicae **67** (1982), 261-274.
- [W2] P. M. WONG, *On Umbilical Hypersurfaces and Uniformization of Circular Domains*, in: *Complex Analysis of Several Variables*. Amer. Math. Soc. Proc. Sympos. (Madison 1982), Pure Math. **41** (1984), 225-252.
- [W3] P. M. WONG, *On the uniformization of parabolic manifolds*, in: *Contributions to Several Complex Variables*, A. Howard and P. M. Wong, eds., Aspects of Math. E9. Braunschweig: Friedr. Vieweg and Sohn, 1986, pp. 329-353.
- [W4] P. M. WONG, *Complex Monge-Ampère Equation and Related Problems*, in: *Complex Analysis III* (College Park, MD, 1985-86). New York: Springer-Verlag Lecture Notes in Math. **1277** (1987), 303-330.

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Department of Mathematics
Texas Tech University
Lubbock, Texas 79409-4319

Department of Mathematics and Computer Science
Wabash College
Crawfordsville, Indiana 47933

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