Area Without Integration: Make Your Own Planimeter

Robert L. Foote
Wabash College

Ed Sandifer
Western Connecticut State University

Introduction

Clay tablets from Mesopotamia and papyri from Egypt provide evidence that work with area has been part of mathematics since its early history. These Ancients knew how to find areas of squares, circles, triangles, trapezoids, and a number of other shapes for which we no longer have names.

Like many other physical quantities, we usually measure area indirectly. That is, we measure something else, such as lengths, a radius, or angles. Then we do some calculations to find area based on appropriate formulas. The object determines the formula we use and the measurements we make.

There are a number of methods to measure volumes. Some of them are indirect, as for areas, involving linear and angular measurements and the use of formulas. Interestingly, some of them are more direct. One example is the measurement of liquid when cooking, using calibrated measuring cups. Similarly, the volume of a non-porous solid can be measured by submerging it in water to see how much water it displaces (it helps if the solid doesn't float).

Many instruments are cleverly designed devices that convert a value we want to measure into some scale that we can read directly. For example, thermometers (the bulb type, not the digital ones) convert the volume of the liquid (this used to be mercury, but now it is usually tinted alcohol) in the bulb into a length. When the liquid expands or contracts with the temperature, the calibrations along the tube let us read the temperature that corresponds to volume.

Our goal is to learn how to make and use a simple device called a planimeter that measures area more directly without having to measure lengths or angles or to make any calculations. A second goal is to understand the mathematics behind how planimeters work.

Small Project #1. Find out how various instruments convert quantities into measurements that we can read directly. There are thousands of different instruments, but you might start with some of the following:

- Barometer — measures air pressure
- Anemometer — measures wind speed
- Speedometer
- Protractor
- Pitot tube — measures speed of an airplane through the air
- Volt meter
- Bathroom scales

- pH meter
- Gasoline pump
- Tire pressure gauge
- Odometer

There are "digital" models of many instruments available. Usually, the digital models work the same as the analog models, but they have a "digitizer" built in to convert the analog reading into a digital value. This gives the illusion of accuracy and makes it harder to figure out what the instrument is actually doing, so we recommend doing this project with analog instruments rather than digital ones.

This project can be a minor homework assignment by having students give a one- or two–line description of how each instrument works. You can build extra credit into the assignment by having students collect as many different instruments as they can find. It can be a more substantial project if students make posters or web pages showing how the instruments work.

Map readers or map distance measurers

Before we understand area, we should make sure we understand length. We can measure straight lengths directly with a ruler. We can use a tape measure to measure around corners and across certain curved surfaces. We can measure more twisted curves indirectly with a string by running the string along the curve and then measuring the string. We could mark the string in order to use it to make direct measurements.

All of these techniques become more difficult if the length to be measured is longer than the device used to measure it. Nobody would want to measure a distance of a mile (or kilometer) or more with a string or a meter stick. Instead, there is a device called a surveyor's wheel, a wheel of known circumference, usually either a meter or three feet, with a counter attached. The counter measures the distance we roll the wheel by counting the number of times the wheel turns. This is also how the odometer on an automobile and a "cyclometer" for a bicycle work.

Cyclometers and odometers work well for measuring long curves. For measuring short distances, such as a path on a map, there is a device called a map reader or a map measurer. It works similarly to a surveyor's wheel. Figure 1 shows a photograph taken from an eBay auction [4]. Its description follows.

> Here is an old instrument used in World War 2 for measuring distances on maps. This precision instrument was manufactured by the Hamilton Watch Company which is one of the oldest and finest watch companies ever. This instrument was manufactured by the Allied Products Division of Hamilton Watch Co. between the years of 1941 and 1957 for the military, then for industrial use afterwards. This one is the "331" model, and is used for distance measurement on maps by using the small roller wheel on a map from point A to point B. This instrument is approximately 5 inches from tip to tip. The gauge is graduated in inches and centimeters.

Figure 1. Map Distance Measurer.

These devices are based on a simple idea. You roll the little wheel (on the right side of the picture) along the route on the map. The device mechanically keeps track of how much the wheel turns. This is recorded on a scale, which can then be read. This will be a fundamental idea in understanding how a planimeter works.

When using a map reader, it is important to keep the little wheel aligned with the path being measured. Otherwise, the wheel slides sideways a bit and does not measure the full length of the path.

Herein lies a theorem. If the wheel does not stay aligned with the path, then what does it measure? It measures just the component of the path that is parallel to the alignment of the wheel. Here is a more precise statement.

Theorem 1 (Map Reader Theorem) *If a map reader rolls along a curve aligned at a constant angle θ to the direction of the curve, then the distance measured by the map reader equals the length of the curve times* $\cos\theta$*. In symbols,* $d = L\cos\theta$*, where d is the distance measured by the map reader (the amount the wheel rolls) and L is the length of the curve.*

It is easy to imagine moving the map reader at a nearly fixed angle to the direction of the curve, due to the way you hold it. It takes practice to get accurate results. Say, for example, that you measure a path that is 20 inches long, and you consistently hold the map reader 7° off from the direction of the curve. Instead of reading $d = 20$ in, it would read $d = 20 \times \cos 7° \approx 20 \times .9925 = 19.85$ in.

Of course the angle between the direction of the wheel and the direction of the curve will not generally be constant. In this case it takes calculus to give a precise description of how much the wheel turns—it is $d = \int \cos\theta \, ds$, where θ can now be a variable angle between the directions of the wheel and the curve, and ds represents the length of an infinitesimal piece of the curve. (This just means that the formula $d = L\cos\theta$ is applied to very short parts of the curve and then the results are added up.) Fortunately you do not need calculus to understand the basic idea: if the wheel is not aligned with the curve, the amount the wheel rolls is something less than the length of the curve. The amount that the measurement is off depends on how bad the alignment is.

Here is an example where the angle between the directions of the wheel and path is not constant in which it is pretty clear how far the wheel rolls. In Figure 2, the wheel of a map reader follows the pictured curved path, however, the user keeps the wheel aligned straight “north’’ on the map instead of aligning it with the path. In this situation, the map reader will always record the distance it is north of the starting point, ignoring the east and west parts of the motion. This is true even if part of the curve has a southerly component, since on this part of the curve the wheel will roll backwards and the distance is subtracted. If we have a coordinate system with the initial point as the origin, then the distance measured by the wheel is the y-coordinate of the final point. Turning the wheel 90° to the right would make the map reader measure the x-coordinate or how far east it has moved from the initial point.

One is tempted to say that the map reader measures “positive’’ distance when it rolls forward and “negative’’ distance when it rolls backwards, but this can be confusing since distance is inherently positive.

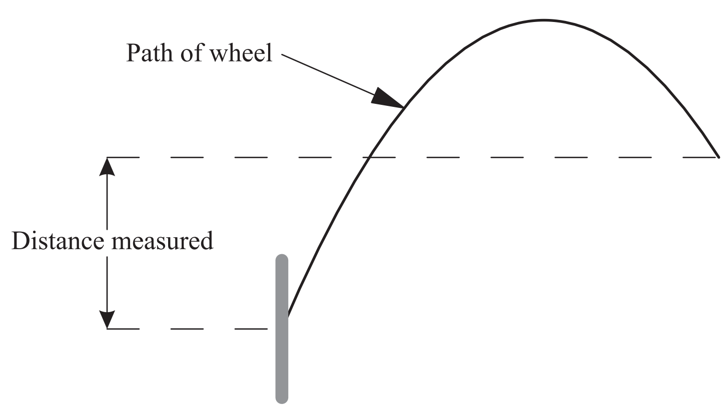

Figure 2. Map reader wheel measuring a y-coordinate.

Nevertheless, this is a useful notion, which we will call “signed’’ or “directed” distance. Thus the map reader measures signed distance, which is the distance it rolls forwards minus the distance it rolls backwards.

Small Project #2. Make a map reader out of TinkerToys®. The wheel of the map reader can be one of the 2’’ spools that lets a stick pass through the center without gripping it. In order to count how many times the measuring wheel turns, it helps if you make one or more marks on the circumference. The marks might be every 30° or 45° around the circumference so you can easily see the number of rotations (including fractions) the wheel rolls. The signed distance is then the number of rotations times the circumference.

Use the map reader to measure a variety of distances, lengths of paths on maps, and *x*- and *y*-coordinates of points. To begin with, note that the measuring wheel is approximately 2’’ in diameter, and so about 6.28’’ in circumference. Start by measuring some line segments of known length. If you get results that are consistently too large or too small, you need to use better value for the circumference. You can get this by rolling the wheel on a piece of paper, marking consecutive places where a point on the circumference contacts the paper, and measuring the distance between these with a ruler.

Lengths and points with coordinates on the order of six to eighteen inches are a good size for this project. This project, especially measuring coordinates along curved paths, should give students a good sense of how the main moving part of a planimeter works. It should take only about 20 minutes.

Measuring area: planimeters

How do you find the area of an irregular region for which there is no formula, such as the area of a lake on a map? You use a planimeter—a mechanical instrument that measures areas. The user traces the boundary of a region with the “tracer point’’ of the planimeter. Upon returning to the starting point, the area of the region can be read from a dial on the planimeter. We will consider two types of planimeters in this section, polar and linear, and a third type in the last section, the Prytz planimeter. Here are some significant dates in the history of planimeters:

- ca. 1814–1853. Bavarian engineer J. M. Hermann and Italian professor Tito Gonnella independently developed the first planimeters in 1814 and 1824, respectively, but their work was not widely known until much later. (For an animation of a Hermann-style planimeter, see [13].) In 1826 the Swiss engineer Johannes Oppikofer also designed a planimeter. It is unclear whether Oppikofer knew of the earlier inventors’ work. Oppikofer’s design was the first to be widely known as well as the first to be manufactured and marketed, and thus is considered by Henrici to be the “starting-point’’ for those that followed. These early planimeters were mechanically complicated, bulky, awkward to use, and it was difficult to get consistently accurate results with them. Other inventors made attempts to address these problems, but were only partially successful [8].
- 1854. Swiss mathematician and inventor Jacob Amsler developed the polar planimeter and a few years later the linear planimeter. They were mechanically simple, small, portable, easy to use, and accurate. They were so clearly superior to existing planimeters that the older planimeters were quickly made obsolete. Since then a few modifications have been made either to improve accuracy or for specific applications [8]. Nearly all modern planimeters use Amsler’s basic design, and even those that do not use his design still work on the same mathematical principle. We will see how to make and use a polar planimeter shortly.
- 1875. Danish mathematician and cavalry officer Holger Prytz developed the hatchet planimeter as an economical alternative to Amsler’s planimeters. This had many of the advantages of Amsler’s

planimeter (in fact it was mechanically simpler), but it was less accurate [8, 10, 5]. We will see how to make and use a hatchet planimeter in the last section.

- ca. 1980–present. Digital planimeters. These have digital readouts and built-in calculators, but are mechanically the same as Amsler's planimeters and not noticeably more accurate. They are still manufactured and sold (see some of the links on [6]).

Amsler's Polar and Linear Planimeters

In 1854, Jakob Amsler invented the polar planimeter, a brilliant and simple device for measuring the area of a region. At the time he was still a student at the University of Koenigsberg. He made a career for himself manufacturing tens of thousands of them, and inventing and manufacturing related instruments.

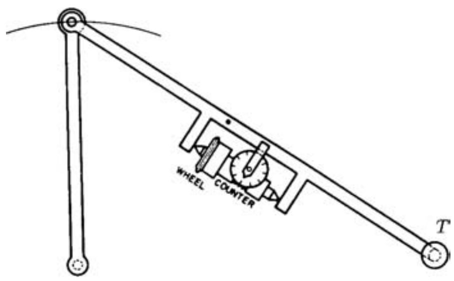

Figure 3. Polar planimeter [9].

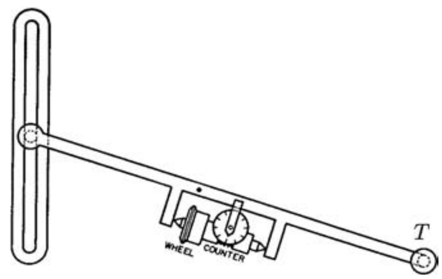

Figure 4. Linear planimeter [9].

Schematic drawings of polar and linear planimeters are shown in Figures 3 and 4. The main part of each is a movable rod, called the tracer arm, with a tracer point at one end (labeled T). A wheel is attached to the rod with its axis parallel to the rod. The wheel is equipped with a scale typically calibrated in square inches or square centimeters. It is similar to a map reader wheel in that it can roll both forwards and backwards, and we will call it the measuring wheel. In a linear planimeter, the end of the tracer arm opposite the tracer point is restricted to follow a linear track, along which it can slide freely. In contrast, in a polar planimeter, the tracer rod is hinged to a second rod, the pole arm, forming an elbow. The end of the pole arm opposite the hinge, called the pole, is fixed so that the pole arm can pivot around it. Consequently the elbow follows the arc of a circle as it moves.

To operate a planimeter, the user selects a starting point on the boundary of the region to be measured, places the tracer point there, and sets the counter on the wheel to zero. The user then moves the tracer point once around the boundary of the region, as shown in Figure 5. The tracer point is typically a stylus or a point marked on a magnifying glass to facilitate the tracing. In a polar planimeter, as the tracer point moves, the elbow at the hinge will flex and the angle between the pole arm and the tracer arm will change. In a linear planimeter, the end of the tracer arm in the track will slide along the track. In both planimeters the wheel rests gently on the paper, partially rolling and partially sliding, depending on how the tracer point is moved. If the pointer is moved parallel to the tracer arm, the wheel slides and does not roll at all. If the pointer is moved perpendicular to the tracer arm, the wheel rolls and does not slide at all. Motion of the pointer in any other direction causes the wheel to both roll and slide. When the tracer point returns to the starting point, the user can read the area from the scale on the wheel.

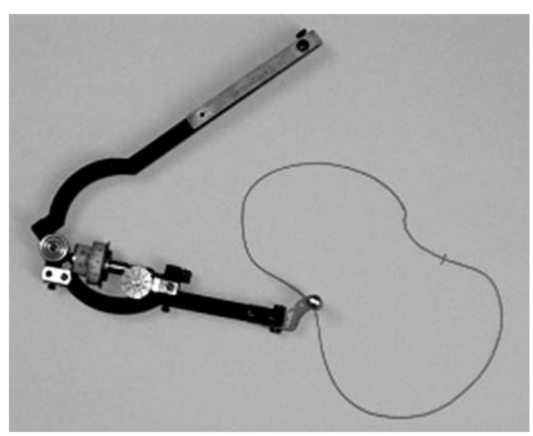

Figure 5. How a planimeter is used.

Like a map reader wheel, the measuring wheel measures a directed distance, namely, the component of its motion that is perpendicular to the tracer arm. The mathematics of how this directed distance can be interpreted as the area of the region traversed by the tracer point is both simple and elegant, and is the basis of how a planimeter works.

Polar and linear planimeters are mechanically simple: polar planimeters need only three moving parts; linear planimeters need only two. Since their invention they have been inexpensive enough that many engineers, architects, surveyors and others could afford to own one. In the electronic age planimeters are less commonly used since the area of a digitized region can be found with an appropriate computer program, but many antique planimeters are still reliable and in service. Modern electronic planimeters are typically expensive, however the electronics are illusory. They are mechanically about the same as their non-electronic ancestors—Amsler would easily recognize them. Furthermore, they are not noticeably more accurate, since their accuracy lies in the skill of the operator, not in the electronics.

Making a polar planimeter

You can easily make a polar planimeter with TinkerToys®. The parts pictured in Figure 6 are from a 2002 Collector's Edition Motorcycle set. It has enough pieces to make three model planimeters.

One polar planimeter uses the following parts (see Figure 6):

4	Spool A	2″ diameter spools that hold sticks tight at the center of the circle
4	Spool C	2″ diameter spools that let sticks pass through the center
2	Long sticks	10.5″ long
2	Short sticks	3″ long
3	Connectors	2″ plastic pieces that connect to spools at the ends and allow sticks to pass through the middle. Some sets do not contain this part.

Assemble it as shown in Figure 7. The figure shows enough parts for *two* planimeters. One set is assembled with the second set placed around it so you can see where each part is used. You can think of the assembly as an arm, with the elbow in the center, the shoulder at the left, and the wrist at the right.

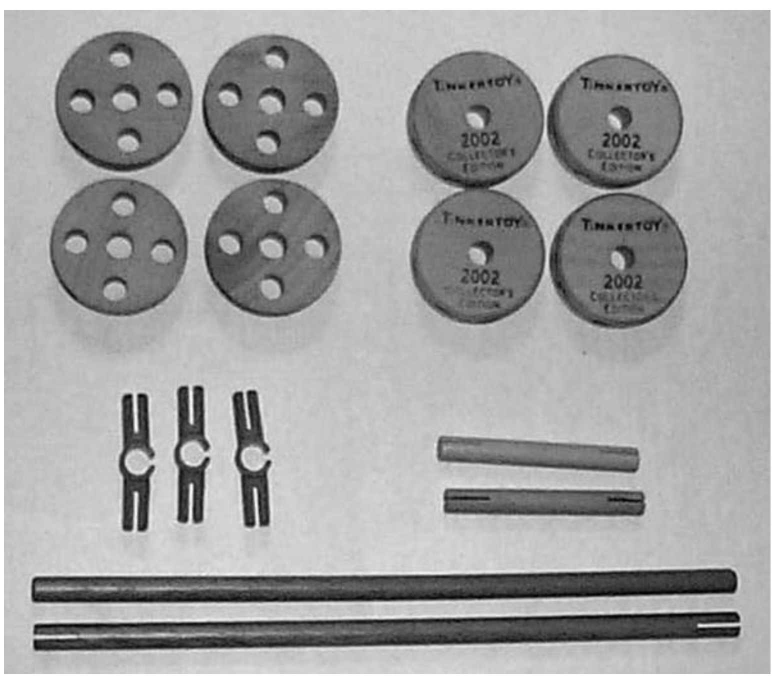

Figure 6. TinkerToys® needed to make a polar planimeter.

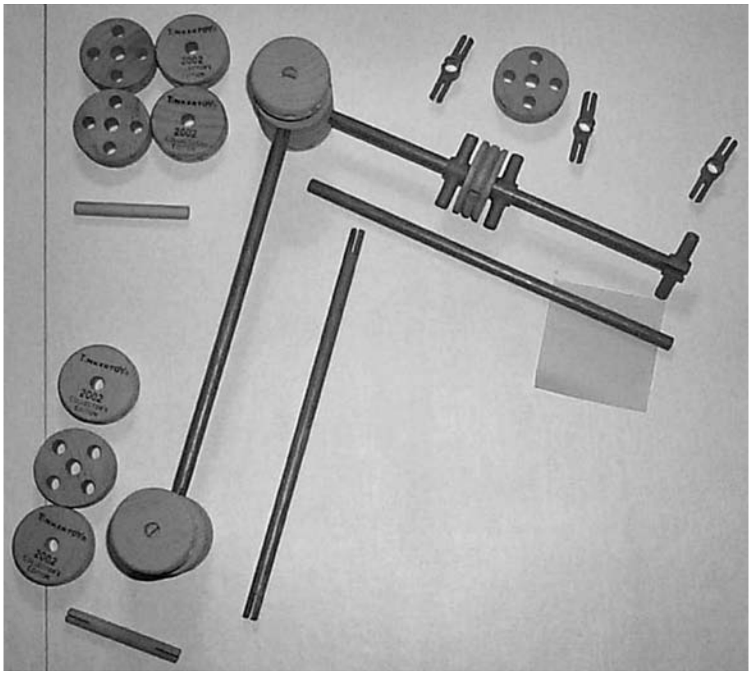
Figure 7. Making a polar planimeter.

The shoulder (which will be the pole) is made of A-Spools connected by a short stick through a C-Spool. The elbow is like the shoulder, but with two C-spools. Join the shoulder to the upper C-Spool in the elbow with a long stick (pole arm). The forearm (tracer arm) is a long stick connected to the lower C-Spool on the elbow. At the center of the forearm, put a C-Spool (measuring wheel), using connectors to keep it as near to the center of the forearm as you can manage, but so that it can still roll smoothly. (If you did Small Project #2 and made marks along the circumference of the wheel, you can use the same wheel for the planimeter. See Small Project #2 for other suggestions.) Align the connectors horizontally so they do not interfere with the rolling. Put the third connector at the wrist, oriented diagonally to the table. This connector is the tracer point. It should be as vertical as possible without reducing the contact of the roller with the surface it rests on. If you like, you can trim the pointer. It should be short enough so it can be oriented vertically, and long enough so it just about touches the surface of the table.

When using the planimeter as described above, you may need to hold the pole to keep it from moving, but be sure to allow the pole arm to pivot around it. After moving the tracer point once around the boundary of the region, as in Figure 8, make a note of the directed distance the measuring wheel rolls. The directed distance is the number of rotations, including fractions, times the circumference. (If you are concerned about the sign of the directed distance, that is, whether it is positive or negative, it depends on which direction you take to be the forward direction of the wheel and which direction you move the tracer point around the boundary of the region. If the net directed distance is negative, you should switch one of these, but not both.)

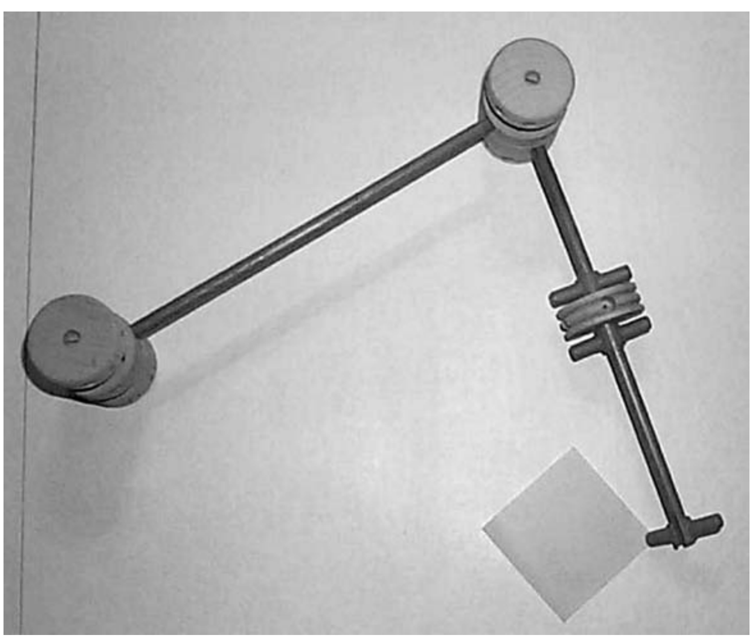
Figure 8. Measuring the area of a square.

The following theorem states the amazing fact that makes this a useful device. Here and later it is convenient to abbreviate “the directed distance the measuring wheel rolls’’ as “the roll of the (measuring) wheel.’’

Theorem 2 (Planimeter Theorem) *The area of a region measured by a polar or linear planimeter equals the* *length of the tracer arm times the roll of the measuring wheel. In symbols, $A_r = Ld$, where A_r is the area of the region, L is the length of the tracer arm, and d is the roll of the wheel.*

The subscript '*r*' is to remind us that A_r is the area of the region being measured (there will be other areas in a moment). Since it will be easier to note the number of rotations made by the measuring wheel, a more useful formula may be $A_r = LnC$, where C is the circumference of the measuring wheel and n is the number of rotations it makes, including fractions.

The proof of the Planimeter Theorem is taken up in the next subsection.

> *Small Project #3.* Use your TinkerToy® planimeter to measure several regions of known area, ranging from 10 to 200 square inches. To do this, you need values for C and L, the circumference of the wheel and the length of the tracer arm, respectively. For C, see the comments in Small Project #1. For L, you can start with 10.5″, the length of the long sticks. If you get results that are larger or smaller than the true area by a consistent percentage, you need to use better values for L and C. The length L of the tracer arm should be the distance on the table between the tracer point and the center of the elbow joint. After calibrating your planimeter, use it to measure some regions of unknown areas. To check your results, compare them to those obtained by others. If you measure the area of a lake, state, or country on a map, convert your result (in square inches) to square miles by multiplying by the *square* of the length scale on the map. For example, if the scale on the map is 15 miles to an inch, you should multiply by $15^2 = 225$, that is, one square inch on the map represents 225 square miles. A lake measuring five square inches on such a map has an area of $5 \times 225 = 1125$ square miles. To check your results, you may be able to find the area in an atlas or almanac.

On a commercial planimeter, the measuring wheel has a scale attached to it, as seen in Figure 9. You do not need to use the formula. Instead, the formula is incorporated into the device, and you read the area directly from the scale. Planimeters are set to measure either in square inches or square centimeters. On some, the length of the tracer arm is adjustable, which allows you to make measurements in either system of units. You can even set it to measure in square miles when using a map with a particular scale.

A linear planimeter is used in exactly the same way. The mechanical difference between the two types is that the "elbow" (the end of the tracer arm opposite the tracer point) of a linear planimeter moves along a straight line, whereas in a polar planimeter it moves along a circle.

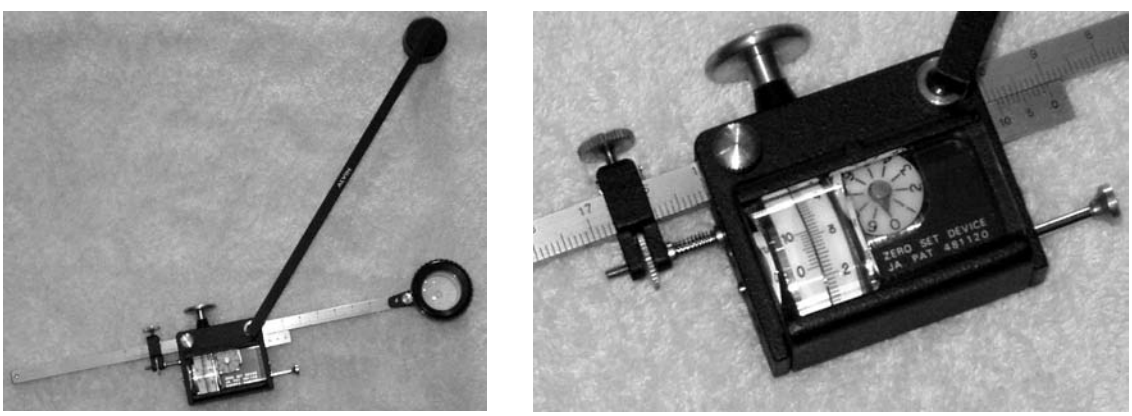

Figure 9. Commercial planimeter and the scale on its measuring wheel.

How Planimeters Work: Proof of the Planimeter Theorem

This seems too simple to work, or, if it works, it seems like it must give only an approximation. However, a planimeter gives exact results, if it is exactly manufactured and used correctly. The proof is based on two simple theorems (Theorems 5 and 6 below) about the area swept out by a line segment moving in a plane, which are significant results in their own right.

Theorem 3 (Moving Segment Theorem [2, p. 295; 3, p. 451]**)** *The area A swept by a straight line segment of length L as it moves from* P_1Q_1 *to* P_2Q_2 *(Figure 10) is given by* $A = Ld$*, where d is the component of the distance the midpoint of the segment moves perpendicular to the segment.*

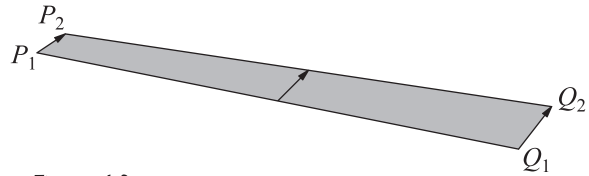

Figure 10. Moving segment sweeping out an area.

It is important to note that d can be (and often is) less than the distance the midpoint moves. To see this, consider the familiar formula for the area of a parallelogram, $A = bh$, where b is the length of the base and h is the height. If a moving segment of length b moves from one base to the other, it sweeps out the parallelogram. The actual distance moved by the midpoint is the length of the other sides of the parallelogram, but the component of its distance perpendicular to the base is the height h. Thus we have $L = b$ and $d = h$, and so the formula for area given by the theorem agrees with that for the area of the parallelogram.

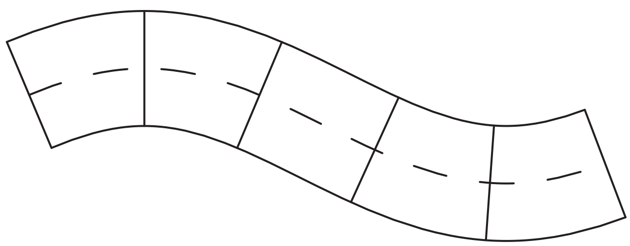

Figure 11. Ribbon Theorem.

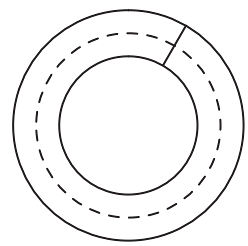

Figure 12. Annulus.

Some readers may be familiar with another special case of Theorem 3, sometimes called the "Ribbon Theorem,'' illustrated in Figure 11.

Theorem 4 (Ribbon Theorem) *The area of a curved region of constant width equals the width of the region times the length of its median curve (the curve described by the midpoints of its widths).*

A special case of this theorem is the well-known formula for the area of an annulus. Let L be the difference between the radii of the circles (shown as a line segment in Figure 12), let C be the circumference of the median circle (dotted line). Then the area of the annulus is $A = LC$. A bit of elementary algebra and geometry shows that this is equivalent to the "obvious'' formula, $A = \pi R^2 - \pi r^2$, where R and r are the larger and smaller radii, respectively.

You have probably already noticed that the formulas in Theorems 2 and 3 are essentially the same, but their interpretations are different. To begin making the connection, note that the d in Theorem 3, awkwardly described as the component of the distance the midpoint moves perpendicular to the segment, is *exactly* the signed distance recorded by a measuring wheel mounted at the midpoint of the tracer arm with its axis parallel to the segment, just as the wheel is attached to the tracer arm of the planimeter. (In the next section we see what happens if the wheel is not mounted at the midpoint.) We note that for Theorem 3 to be true, areas that are swept more than once must count as many times as they are swept. Furthermore, if part of the segment backtracks, that portion of the area must be counted negatively. Like distance, area is not negative, but we can consider a new notion of *signed area* relative to the moving segment. The segment has a forward direction (the direction in which the signed distance measured by the wheel is positive). Signed area swept in that direction is positive. Signed area and signed distance in the opposite direction are negative. Thus Theorem 3 is really about signed area. It can be restated more simply as follows.

Theorem 5 (Moving Segment Theorem, restated) *Suppose a line segment of length L is equipped with a measuring wheel at its midpoint with its axis parallel to the segment. If the segment moves from one position in a plane to another, the signed area swept out by the segment is given by $A = Ld$, where d is the signed distance recorded by the wheel during the motion.*

So now the *d* in Theorem 2 is the same as the *d* in Theorem 3. What about the areas? Note that the overall area covered by the tracer arm is typically much larger than the area being measured. However, the *signed* areas are the same. This follows from the following Area Difference Theorem.

Theorem 6 (Area Difference Theorem) *If the endpoints of a moving line segment each move around closed curves, both in the same direction, then the signed area swept by the segment is equal to the difference between the two areas traversed by the endpoints. In symbols, the signed area is $A = A_r - A_\ell$, where A_r and A_ℓ are, respectively, the areas of the regions traversed by the right and left endpoints (Figure 13, lower right).*

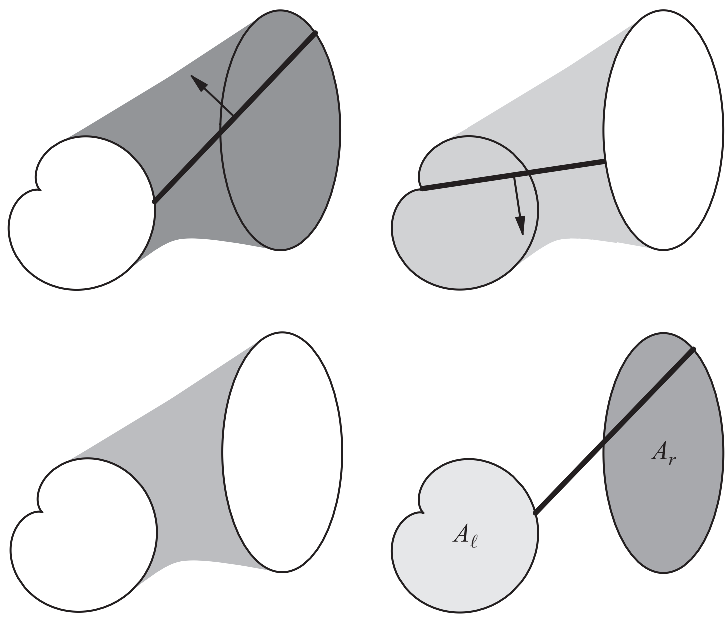

Figure 13. Intuitive proof of the Area Difference Theorem [6].

(Now the subscript '*r*' stands for 'right'! Not to worry—we will always take the region measured by a planimeter to be the one on the right.) Intuitively, here is why this is true. The total signed area swept by the segment is equal to the area swept in the positive direction (Figure 13, upper left) minus the area swept in the negative direction (Figure 13, upper right). The positive area has two parts, namely A_r, which is encircled by the right endpoint, and that between the two regions (Figure 13, lower left). Likewise, the negative signed area has two parts, A_ℓ, encircled by the left endpoint, and that between the two regions. When we add to get the total signed area swept by the segment, the parts in common cancel, leaving the signed area swept by the segment as the area encircled by one end minus the area encircled by the other end, that is, $A_r - A_\ell$. For an animation of this, see Robert Foote's website [6]. Bruce Atwood [1] gives a rigorous proof using Green's Theorem.

Combining Theorems 5 and 6, we have $A_r - A_\ell = A = Ld$. If the moving segment is the tracer arm of a planimeter, then A_r is the area enclosed by the tracer point of the planimeter, that is, the area of the region being measured, and A_ℓ is the area enclosed by the "elbow." But, as noted earlier, the linkage of a polar planimeter is designed so that the elbow never leaves the arc of a circle, hence, properly used, it will encircle no area, that is, $A_\ell = 0$. Hence, the area enclosed by the tracer point equals the signed area swept by the tracer arm. The situation for the linear planimeter is similar—the end of the tracer arm that moves along the linear track encloses no area, so again $A_\ell = 0$. In both cases we have that $A_r = Ld$. This completes the proof of Theorem 2, the Planimeter Theorem.

It Does Not Matter Where the Wheel Is

Up to this point it has been assumed that the wheel of the planimeter is located at the midpoint of the tracing arm. You may have noticed, however, that on most commercial planimeters, the wheel is generally *not* at the midpoint. In this section we will see that the wheel can, in fact, be located at any point along the tracing arm.

From the Moving Segment Theorem we have the formula

$$A = Ld, \tag{1}$$

where A is the signed area swept out by a motion of the tracer arm, d is the corresponding roll of the wheel at the midpoint, and L is the length of the tracer arm. Relocating the wheel changes this formula, but not the formula in the Planimeter Theorem. Again, the formulas are the same, but they have different interpretations. The formula in the Moving Segment Theorem is true regardless of the segment's position, whereas the formula in the Planimeter Theorem is valid only after the region has been measured. We need to determine how locating the wheel at another point changes formula (1).

Small motions of the tracer arm come in two independent types. The general small motion is a combination of these. The first type is translational, when the planimeter's initial and final positions are parallel to each other. In this case the signed area swept out is given by (1), since this is just the formula for the area of a parallelogram, as noted in the previous section. This is valid no matter where the wheel is located along the tracer arm.

The second type of motion is when the tracer arm rotates about the wheel, that is, the wheel does not roll, and the planimeter's initial and final positions make some angle θ to each other (see Figure 14). Two circular sectors are formed. It is important to note that they are swept out in opposite directions, and so one of them contributes positively to the signed area and the other contributes negatively. If the rotation is clockwise, the signed area of the left sector is positive and that of the right sector is negative—the opposite is true for a counterclockwise rotation. Note that if the wheel is mounted at the midpoint of the tracer arm, these two circular sectors are congruent, and so their signed areas cancel, which makes this special case easier.

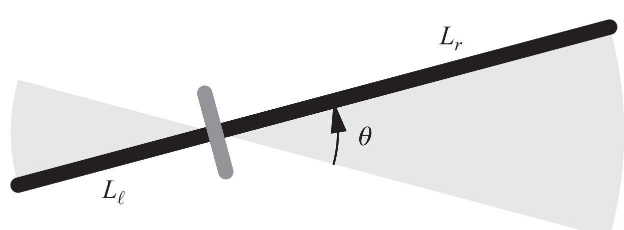

Figure 14. Rotation about the wheel [6].

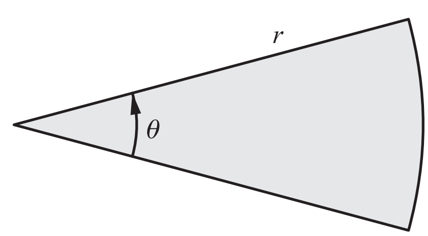

Figure 15. Circular sector.

Suppose that the wheel is located along the tracer arm so that the length of the arm to the left of the wheel is L_ℓ and the length of the arm to the right of the wheel is L_r. Then L_ℓ and L_r are the radii of the two circular sectors. Recall that a circular sector of radius r and central angle θ (see Figure 15) has area given by

$$A = \frac{1}{2}r^2\theta \qquad \text{or} \qquad A = \frac{\pi}{360}r^2\theta.$$

Here and in what follows, the first formula is used if the angle θ is measured in radians; the second formula is used if the angle θ is measured in degrees.

Assume for the moment that the small rotation of the tracer arm about the wheel in Figure 14 is counterclockwise. The signed area swept out is the difference between the areas of the circular arcs. Applying the previous formula we have

$$A = \frac{1}{2}L_r^2\theta - \frac{1}{2}L_\ell^2\theta = \frac{1}{2}\left(L_r^2 - L_\ell^2\right)\theta \qquad \text{or} \qquad A = \frac{\pi}{360}L_r^2\theta - \frac{\pi}{360}L_\ell^2\theta = \frac{\pi}{360}\left(L_r^2 - L_\ell^2\right)\theta. \tag{2}$$

Note that even though the angle θ is positive, the signed area A can be positive or negative, depending on whether L_r is bigger or smaller than L_ℓ. If we assume the standard convention that counterclockwise angles are positive and clockwise angles are negative, then this formula is also correct for clockwise rotations—if the rotation is changed from counterclockwise to clockwise, both θ and A are multiplied by –1, and so equality is maintained in (2).

We have seen that (1) and (2) give the signed areas swept out by the two types of small motions of the tracer arm. The total signed area swept out is the sum of these:

$$A = Ld + \frac{1}{2}\left(L_r^2 - L_\ell^2\right)\theta \qquad \text{or} \qquad A = Ld + \frac{\pi}{360}\left(L_r^2 - L_\ell^2\right)\theta. \tag{3}$$

In fact, this formula is valid for large motions as well.

Theorem 7 (Improved Moving Segment Theorem) *Suppose a line segment of length L is equipped with a measuring wheel with its axis parallel to the segment. If the segment moves from one position in a plane to another, the signed area swept out by the segment is given by formula* (3).

Two observations are in order. First, note that if the wheel is at the midpoint of the tracer arm, then $L_\ell = L_r$ and (3) reduces to the special case (1) of the previous section, as it should. Second, the two terms of the sum in (3) are complementary to each other in a mechanical way: Ld applies when the wheel rolls and the tracer arm does not rotate, whereas $\frac{1}{2}\left(L_r^2 - L_\ell^2\right)\theta$ applies when the tracer arm rotates and the wheel does not roll.

Now suppose the planimeter is used to measure an area. The Area Difference Theorem applies and we have $A = A_r - A_\ell$. As in the previous section, A_ℓ is zero because the left endpoint of the arm does not go around any area, and so A equals A_r, the area of the region being measured. In addition, note that the tracer arm does not make a full rotation; its direction simply oscillates around its initial direction. It follows that the net change in angle, θ, is zero. Plugging these into (3) we get

$$A_r = Ld.$$

This is the same result as in the Planimeter Theorem. Since it does not involve L_ℓ and L_r, it follows that the wheel can be located anywhere along the tracer arm. The area of the region is the product of the length of the tracer arm and the amount the wheel rolls, and this is what is read on the wheel's scale.

Commercial planimeters are invariably used by tracing the region clockwise, as opposed to counterclockwise. This makes the wheel roll in the opposite direction. If you have access to one, you will note that the scale on the wheel is calibrated so that it *decreases* in what we have used as the forward direction.

The Prytz Hatchet planimeter

We are now in a position to understand the operation of a very simple and curious type of planimeter, known as a hatchet planimeter. It has no moving parts! It is surprising that it measures anything at all, but with formula (3) in the Improved Moving Segment Theorem we will be able to show that it measures area, at least approximately, and that the error has a very nice geometric interpretation.

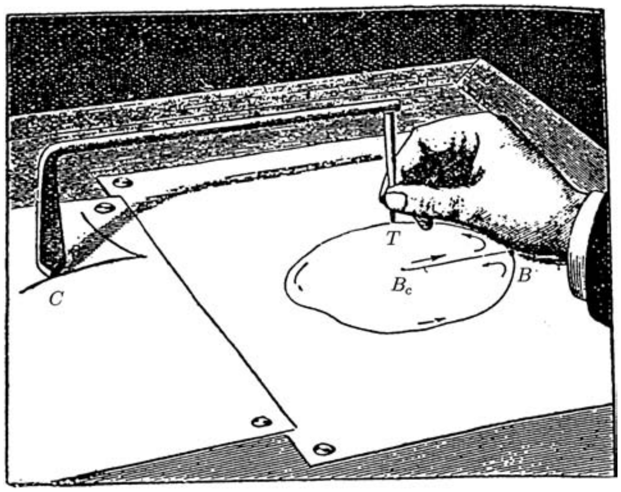

Figure 16. Hatchet planimeter [11].

This type of planimeter was invented in about 1875 by Holger Prytz, a Danish cavalry officer and mathematician, as an economical and simple alternative to Amsler's polar planimeter. It consists of a rod with its ends bent at right angles to its length. One end, labeled T in Figure 16, is sharpened to a point. The other end, labeled C, is sharpened to a chisel edge parallel to the length of the rod. The chisel edge is slightly rounded, making it look somewhat like a hatchet (hence the name). Prytz referred to it as a "stang planimeter,'' "stang'' being the Danish word for "rod.''

The first author had a Prytz planimeter made from a used chemistry ring stand in the Wabash College maintenance shop. It is also easy to make one out of TinkerToys® or wooden dowels, as in Figure 17. Use two 1/4'' dowels for the vertical pieces. Sharpen one to a point in a pencil sharpener. The chisel edge can be fashioned in the other dowel by carving it. Alternatively, cut a thin slot in the end and wedge into it a small piece of sheet metal or the tip of an old table knife. Even a sturdy piece of thin cardboard will work (cut a small circle from an old playing card). We have even seen one in which the chisel edge is a circular, rotating, pizza slicer with its handle extended by a dowel. The chisel edge does not need to be terribly sharp, but it does need to be rounded and stiff enough that it will track correctly. It is important to be able to hold and guide the device with a relaxed grip. It helps if the center of gravity is low. The vertical pieces should be no longer than necessary and the horizontal bar should be lightweight, say a 3/8'' dowel or other piece of wood. A small weight (a coin, for instance) can be added just above the chisel edge to help it track better (see Figures 17 and 19).

Figure 16 illustrates how the device is used. The end sharpened to a point is the tracer point T. Before beginning to trace the boundary of the region to be measured, note the initial location C_i of the chisel edge.

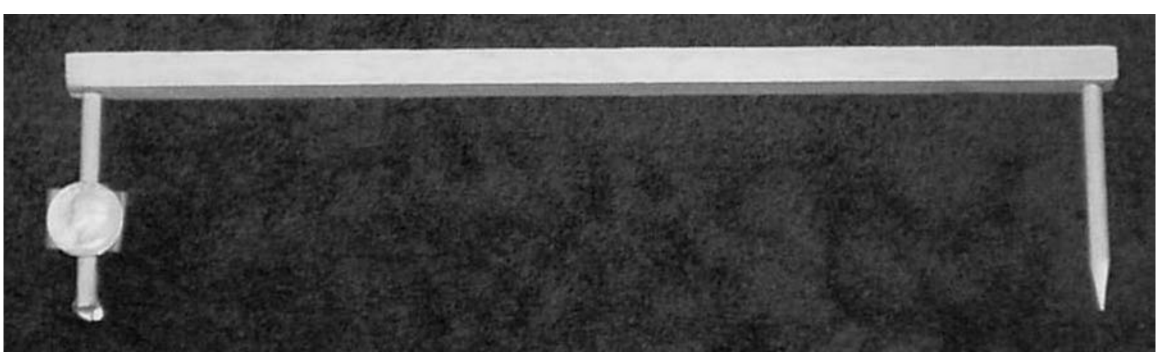

Figure 17. Homemade hatchet planimeter.

Then trace the boundary of the region, taking care not to apply any torque to the rod. Using a relaxed grip is important here. The chisel edge will track better if it drags on a piece of paper as opposed to a smooth counter top. The chisel edge will describe a zig-zag path that is always tangent to the direction of the rod (see Figures 16 and 18). After traversing the boundary of the region, make a note of the final location C_f of the chisel edge. Measure the distance between C_i and C_f and call it D. Measure the length L of the rod, that is, the distance between the tracer point and the point where the chisel edge contacts the paper. Then the area A_r of the region is given, at least approximately, by their product,

$$A_r \approx LD. \tag{4}$$

What could be simpler?

This may seem too good to be true, and in some sense it is. Of course, *every* measurement is an approximation, but the main source of error in the use of a standard polar planimeter is due to the errors made by the user (not following the curve accurately or not reading the scale correctly). On the other hand, the error in the approximation made by a hatchet planimeter is inherently part of the device. The error can be significant, even if the user has a steady hand and measures D and L accurately. With the background of the previous sections, we can understand this approximation and the nature of the error, which can lead to a more accurate use of the instrument.

> *Small Project #4.* Make a hatchet planimeter. Use it and formula (4) to measure the areas of several regions, and compare your results with those you obtained with the polar planimeter. You may want to repeat this project after reading the rest of this section to understand how to minimize the error inherent in the hatchet planimeter.

The hatchet planimeter rod is the analog of the tracer arm in a polar or linear planimeter; it is the moving line segment that sweeps out signed area. As in the previous sections, take the tracer point to be the right end point. Then the chisel edge is the left end point. We would like to apply the Improved Moving Segment Theorem and the Area Difference Theorem, however there are two obstacles, the most obvious of which is that there is no wheel. The second is that the chisel edge does not traverse a closed curve, which is required by the Area Difference Theorem.

We overcome these obstacles with a "thought experiment." Consider using the hatchet planimeter to measure the region shown in Figure 18. The initial and final positions of the rod are shown, as well as the path of the chisel. After the tracer point returns to its starting position, imagine holding the tracer point fixed and rotating the rod from its final position back to its initial position. The chisel edge follows a circular arc from C_f to C_i. The radius of the arc is L, the length of the rod, and its center is the base point B where the tracing starts and stops. Now the chisel edge has gone around a closed curve, and the Area Difference Theorem applies. Actually we need a modification of the Area Difference Theorem because the curve intersects itself.

Theorem 8 (Improved Area Difference Theorem) If the endpoints of a moving line segment each move around closed curves, then the signed area swept by the segment is $A = A_r - A_\ell$, where A_r and A_ℓ are, respectively, the signed areas of the regions traversed by the right and left endpoints.

It is the same as before, but with a more sophisticated interpretation. The area of a region is taken to be signed (positive or negative) depending on the direction its boundary is traversed. The area is taken to be positive when the boundary is traversed counterclockwise; it is taken to be negative when the boundary is traversed clockwise.

The tracer point has gone around its region counterclockwise, and so A_r is simply the area we are measuring. The chisel edge, however, has gone around two regions that are roughly triangular (shaded in Figure

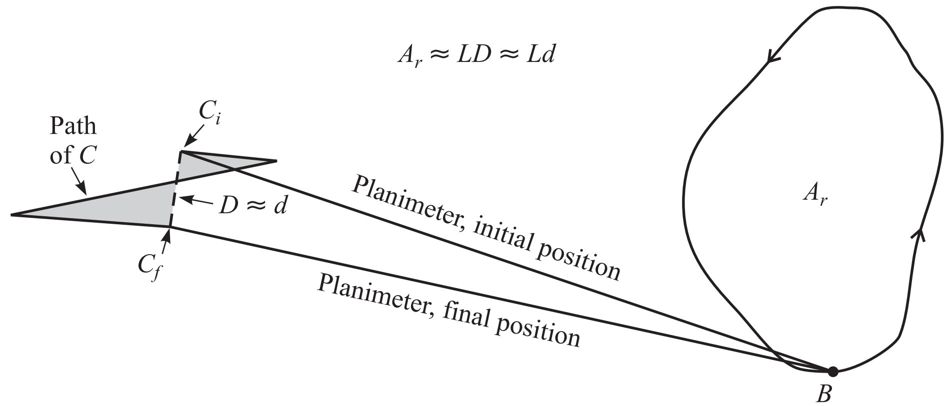

Figure 18. Analysis of the zig-zag path of the chisel [6].

18). In particular it has gone around the upper one (with C_i as a vertex) clockwise and the lower one (with C_f as a vertex) counterclockwise. Let A_{cw} and A_{ccw} be, respectively, the areas of the upper and lower triangular regions. Then A_ℓ, the signed area enclosed by the path of the chisel, is $A_\ell = A_{ccw} - A_{cw}$. By the Improved Area Difference Theorem, the signed area swept out by the rod is

$$A = A_r - A_\ell = A_r - (A_{ccw} - A_{cw}). \tag{5}$$

In a more complicated example there could be many triangular regions, in which case A_{cw} would be the total area of those traversed clockwise, and similarly A_{ccw} would be the total area of the those traversed counterclockwise.

Continuing our thought experiment, imagine that the chisel is replaced by a wheel with its axis parallel to the rod as usual. How much does the wheel roll? While the tracer point goes around the boundary of the region being measured and the wheel follows the zig-zag path, *the wheel does not roll at all*. This is because the direction of motion is parallel to the rod, and this is the direction in which the wheel only slides. On the other hand, when the planimeter rod is rotated from its final position back to its initial position, the amount it rolls is the length of the circular arc from C_f to C_i. As in the previous sections, denote the roll of the wheel by d.

We can now apply formula (3) from the Improved Moving Segment Theorem. Just as in the previous section, since the planimeter (in the thought experiment) returns to its initial position and does not make a full rotation, the net change in angle, θ, is zero. Plugging this and (5) into (3), we have

$$A_r - (A_{ccw} - A_{cw}) = Ld.$$

Solving for the area we are interested in, we get

$$A_r = Ld + (A_{ccw} - A_{cw}).$$

If we think of the product Ld as an approximation for A_r, then this formula shows that the error made by the approximation is the quantity in parentheses, that is,

$$A_r \approx Ld \quad \text{and} \quad \text{error} = A_{ccw} - A_{cw}. \tag{6}$$

Note that d is approximately the distance D between C_f to C_i that we measured earlier in (4). The use of D instead of d (which is how (4) follows from (6)) changes the error, but not significantly. However this had a significant effect on the history of the instrument, as we will see in a moment.

The analysis leading to (6) shows that the amount of error made is a trade-off between the triangular regions traversed clockwise by the chisel and those traversed counterclockwise. With some experimentation you can convince yourself that the starting position of the planimeter affects this trade-off. With practice you can judge which positions are better than others. In a more detailed analysis, Prytz showed that the error is substantially reduced, although not entirely eliminated, by starting the tracing point at the centroid, or center of mass, of the region, as is suggested in Figure 16 (the centroid is labeled B_c). Draw a line segment from the centroid to some point on the boundary. Start tracing at the centroid, following the line segment out to the boundary. After going around the boundary, follow the line segment back to the centroid. Of course determining the exact location of the centroid is harder than finding the area, but the idea is that an educated guess should reduce the error. Prytz also showed that the error is smaller if a longer planimeter is used. Prytz' paper is quite technical, but its main points are summarized in [5].

The hatchet planimeter has an amusing history. For some unknown reason, Prytz wanted to remain anonymous, at least initially. He published the theory of how the instrument works (considerably more complicated than that presented here) under the pseudonym "Z'' in 1886. Eventually others made design modifications and wanted to take credit for the device. In 1894–96 and again in 1906–07 there ensued a lively debate in the pages of the journal *Engineering* as to the identity of the inventor and the merits of the subsequent modifications [5, 10].

Most of the design changes involved measuring the circular arc distance d between the initial and final chisel locations instead of simply using the straight line distance D. The inventors were under the mistaken impression that this would correct the inherent error of the device. (It does not, as the analysis above shows.) One such modification, by an engineer named Goodman in 1896, is shown in Figure 19. Here a curved scale is incorporated into the planimeter rod. The radius of the curve is the same as the length L of the rod. By placing the curved scale alongside the initial and final chisel locations, one measures d directly. As with the scale on the wheel of a polar planimeter, Goodman incorporated the constant multiplier L so that the product Ld, the approximate area, is read from the scale. Note the weight over the chisel edge to help it track better. The design changes made the instrument more expensive and defeated its purpose, in Prytz's opinion, as an economical alternative to a polar planimeter. In harsh criticism of the other inventors' designs, Prytz advised engineers [12] "rather than use the 'improved stang planimeters,' let a country blacksmith make them a copy of the original instrument.''

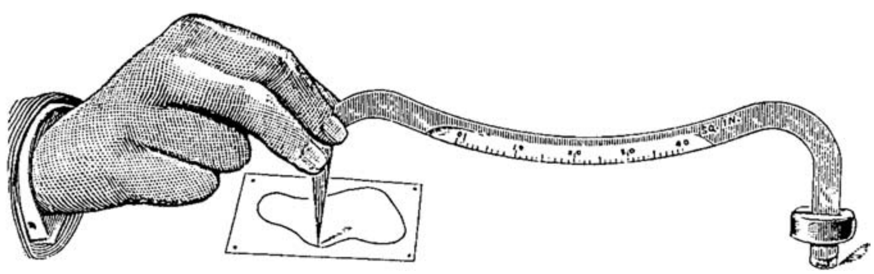

Figure 19. Goodman's version of a hatchet planimeter [7].

Conclusion

This simple analysis of how these planimeters work is due to Henrici [8], who gives a very complete and interesting history of planimeters up through 1893. For more details and references on the history of hatchet planimeters, see [10] and [5]. Another good reference on planimeters and other mathematical devices, which may be easier to find, is the book by Murray [9]. A recent article [14] gives very nice proofs of how polar and linear planimeters work, suitable for students who have studied calculus.

Polar planimeters are readily available on eBay [4], and can typically be purchased for $50 to $100. Linear planimeters appear occasionally and are generally more expensive. Manufactured hatchet planimeters are quite rare and expensive. You might also find a planimeter hiding forgotten in a drawer or closet in a local school or college science department, or in the attic of a retired scientist, architect, or engineer. There are other kinds of planimeters. For further information and pictures, see the first author's web page [6].

Planimeters are still manufactured, and a web search will turn up a few companies that sell them. Many have electronics added, but mechanically they are really just polar and linear planimeters with calculators riding piggyback; Amsler would easily recognize them. The only difference is that the roll of the wheel is read electronically instead of from a scale, and the calculator allows the user to convert between different units, say from square inches to square centimeters, or to square miles based on the scale of a map. Engineering, architecture, and natural resource students are still taught how to use them at many schools, and they are used by many professionals in these and other fields.

Areas are useful in numerous applications. Uses of planimeters range from finding areas of lakes on maps, to finding areas of tumors on x-rays and of leaves and flower petals. The Science Museum in London has a huge polar planimeter with arms that are over three feet long that was used in the leather industry to measure areas of hides.

Small Project #5. Now that you know how planimeters work, here are some related things to explore.

- Make a more accurate polar planimeter than the one you made out of TinkerToys®.
- Make a linear planimeter.
- Determine how the accuracy of a Prytz planimeter depends on its starting orientation. For a given starting point along the boundary of the region, some orientations will minimize the error made in (6).
- Do your science departments have a planimeter tucked away in a drawer? Does your local historical society have a planimeter in its collection? Do they have it on display? Do they know what it is and how to use it?
- Do a web search to find additional uses of planimeters. You might even find the web page in which some researcher in food science has used a planimeter to measure the area of a cookie!

Acknowledgments

This paper began as a handout for a presentation made by the second author in January 2004 at an MAA Short Course on the history of mathematical instruments. We would like to express thanks to the editor, Amy Shell-Gellash, who was also one of the organizers of the course, for encouraging us to expand it into a paper.

Figures 3 and 4 are from *Mathematical Machines*, Vol. 2, *Analog Devices* by Francis J. Murray [9], Copyright ©1961 by Columbia University Press. Reprinted with permission of the publisher.

References

1. B. Atwood, *Area Measurement: Planimeters & Green's Theorem*, http://www.attewode.com/Calculus/AreaMeasurement/area.htm, 28 August 2007.
2. R. Courant, *Differential and Integral Calculus*, II. Originally published in 1934 as *Vorlesungen über Differential- und Integralrechnung.* Translated by E. J. McShane. Wiley Interscience, New York, 1988.
3. R. Courant and F. John, *Introduction to Calculus and Analysis,* II, Wiley Interscience, New York, 1974.
4. eBay on line auctions, www.ebay.com.
5. R. Foote, "Geometry of the Prytz Planimeter", *Reports on Mathematical Physics*, **42** (1998) 249–271.
6. ——, *Planimeters*, http://persweb.wabash.edu/facstaff/footer/planimeter/planimeter.htm, 28 August 2007.
7. J. Goodman (pub. anon.), "Goodman's Hatchet Planimeter", *Engineering*, Aug. 21, 1896, 255–56.

8. O. Henrici, *Report on Planimeters*, British Assoc. for the Advancement of Science, Report of the 64th meeting, (1894) 496–523.
9. F. J. Murray, *Mathematical Machines*, Vol. 2, *Analog Devices*, Columbia University Press, New York, 1961.
10. Olaf Pedersen, “The Prytz Planimeter”, in *From Ancient Omens to Statistical Mechanics*, J. L. Berggren and B. R. Goldstein, eds., University Library, Copenhagen, 1987.
11. A. Poulain, “Les Aires des Tractrices et le Stang-Planimètre”, *J. de Mathématiques Spéciales*, Vol. 4, No. 2 (1895) 49–54.
12. H. Prytz, “The Prytz Planimeter”, (two letters to the editor), *Engineering*, September 11, 1896, 347.
13. Würzburger Mathematikausstellungen: Kegel-Planimeter (Würzburger Mathematics exhibition: Cone Planimeter), http://www.didaktik.mathematik.uni-wuerzburg.de/History/ausstell/planimet/gonnella.html, 28 August 2007.

Added in proofs:

14. T. Liese, “As the Planimeter’s Wheel Turns: Planimeter Proofs for Calculus Class”, *The College Mathematics Journal*, 38 (1007) 24–31.