Problem 11

Yang Yang

April 3, 2017

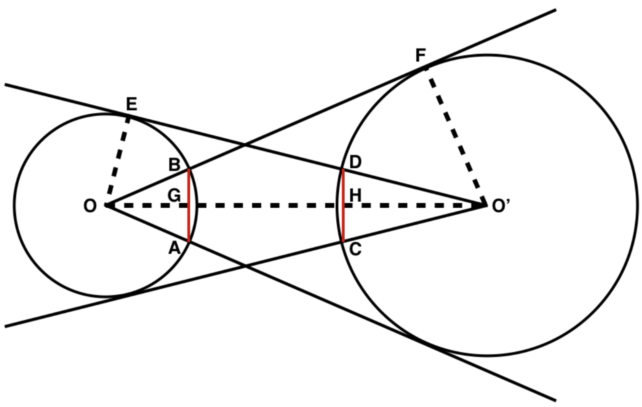
This proof is based on sine rule in these three geometries. Sine rule in Euclidean geometry states if we have a triangle ABC with each angle corresponding to an edge a, b and c, we will have

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}.$$

In spherical geometry and hyperbolic geometry, the sine rule becomes

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)} \qquad \text{(spherical geomertry)}$$
$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)} \qquad \text{(hyperbolic geomertry).}$$

Noticing sine rule in three different geometries only differs by the denominator part, we can rewrite $\sin(A)/a$, $\sin(A)/\sin(a)$ and $\sin(A)/\sinh(a)$ as $\sin(A)/f(a)$, and hence we can replace $f(a)$ to reach the conclusion in any of these three geometries.



When we connect the centers of two circles O and O', OO' becomes a perpendicular bisector the chord AB and CD at G and H due to the symmetry of two circles. Since OF and $O'E$ are two tangent lines, $\angle OEO'$ and $\angle OFO'$ are two right angles. Since $AB = 2BG$, we need to find the length of BG. In $\triangle BOG$, we have $\sin(\angle BGO)/f(OB) = \sin(\angle BOG)/f(BG)$, so $f(BG) = (\sin(\angle BOG)/\sin(\angle OGB))f(OB) = (\sin(\angle BOG)/\sin(\pi/2))f(OB) = \sin(\angle BOG)f(OB)$. Since $\angle BOG = \angle FOO'$, we can find $\sin(\angle FOO')$ in $\triangle OFO'$. In $\triangle OFO'$, we have $\sin(\angle FOO')/f(O'F) = \sin(\angle OFO')/f(OO') = \sin(\pi/2')/f(OO')$, so $\sin(\angle FOO') = f(O'F)/f(OO')$. If we let $OG = r$, the radius of the small circle, $O'F = R$, the radius of the large circle, and $OO' = d$, the distance between two centers of two circles, then we have $f(BG) = f(r)f(R)/f(d)$.

Similarly, we can also find $f(DH)$ from $\triangle DHO'$ and $\triangle O'EO$. Followed by the steps $f(DH) = \sin(\angle DO'H)f(O'F) = \sin(\angle DO'H)f(R)$ in $\triangle DHO'$ and $\sin(\angle DO'H) = \sin(\angle EO'O) = f(OE)/f(OO') = f(r)/f(d)$ in $\triangle O'EO$, we have $f(DH) = f(r)f(R)/f(d)$.

Therefore, we showed $f(DH) = f(BG)$, which indicates $DH = BG$ in all three different geometries.