Problem 10

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In order to investigate the possible sites for the target ball based on the initial cue ball position x, suppose if the cue ball is able to hit the target ball after bouncing on the south and west cushion at some initial hit angle, we take away the target ball and hit the cue ball with the same initial angle to see where the cue ball hit on the north cushion (see the red path in the figure below). If we label the distance between the left corner O to where the cushion ball hit the south cushion A as y. Then based from similar triangles ΔAOB and ΔAHG we can get the length of OB as $OB = \frac{5y}{2(10-x-y)}$. Since we know $OC = 5$, we have $BC = 5 - \frac{5y}{2(10-x-y)}$, and hence from similar triangles ΔAOB and ΔBCD, we get $CD = 2(10 - x - y) - y$.

Then we can consider the extreme case where $CD = 0$ and $AO = 0$. When $CD = 0$, we have the max $y = \frac{2}{3}(10 - x)$. When $y = 0$, we have max $CD = 2(10 - x)$. Noticing when $x < 5$, $CD > 10$, we can treat the cue ball virtually hit the north cushion at I, although it will hit the east cushion before it. Nevertheless, it doesn't affect the result if the target ball is along HF, we can have the cue ball hit the target ball.

If we set up Cartesian coordinate at O, with OG direction as $x-$axis and OC direction as $y-$axis. We will be able to hit the target ball being placed between $B: \ (0, \frac{5y}{2(10-x-y)})$ and $D: \ (2(10 - x - y) - y, 5)$. With the extreme condition considered above, we can find the possible region for the target ball is the region swept by BD during the change of y. Hence in the figure below, the trapezoid region $OCEF$ is the possible region. Noticing when $x > 5$, the cue ball will no longer be able to directly hit the east cushion without hitting the north cushion, the possible region becomes a triangle ΔCOF under this condition. Therefore, we can find the length of EF from simple geometry as $EF = 5 + \frac{25}{x-10}$, and thus the area of the possible region is $5(10 + \frac{25}{x-10})$ for $x \leq 5$. When $x > 5$, we have $CF = 2(10 - x)$, and hence the area of the possible region is $5(10 - x)$. Therefore, we have the area of the possible region as a function of x

$$A(x) = \begin{cases} 5(10 + \frac{25}{x-10}) & x \leq 5 \\ 5(10 - x) & x > 5 \end{cases}$$