Problem 9

Yang Yang

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Given the function $f(f(n)) = -n$, we know $f(f(f(f(n)))) = n$, so we know f has a s symmetry of order 4, which means when we apply the operation four times, it goes back to itself. Since we have the operation switching between positive and negative as a symmetry of order 2, we need to find another operation with a symmetry of order 2 such that we can combine them to achieve $f(n)$ which has the symmetry of order 4. Besides the sign of integer, we notice we can group each number with its even or odd complement to form a pair, so the operation switching between even and odd can be an operation with the desired symmetry. However, zero is a special case for the operation switching between positive and negative, because zero neither belongs to negative numbers nor positive numbers, and hence we will treat zero as a special case in the generalization of $f(n)$, after we generalize $f(n)$ on $\mathbb{Z}^*$ (all the integers excluding zero).

As the principle stated above, every odd number $2n-1$ is paired with an even number $2n$, including their negative pair $-2n+1$ and $-2n$, where $n = 1, 2, 3, \ldots$. Therefore, $\mathbb{Z}^*$ is partitioned into groups with four elements in each group. If we define $f(n)$ mapping through $2n-1$, $2n$, $-2n+1$ and $-2n$, we can have when n is positively odd, $f(n) = n+1$; when n is positively even, $f(n) = -(n-1)$; when n is negatively odd, $f(n) = n-1$; when n is negative even, $f(n) = -(n+1)$. This can be generalize into a simple function

$$f(n) = n(-1)^n - \frac{n}{|n|}.$$

At last, we treat the special case 0 by letting $f(0) = 0$. Therefore we have the function $f : \mathbb{Z} \to \mathbb{Z}$ as

$$f(n) = \begin{cases} n(-1)^n - \frac{n}{|n|} & n \neq 0, \\ 0 & n = 0. \end{cases}$$