Fortnight 8

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Since one and a half chickens lay one and a half eggs in one and a half days, we can conclude that it takes one chicken a day and a half to lay one egg. Thus, if we want 14 eggs, we will need $\frac{14}{c}$ sets of days and a half to lay eggs, where c is the number of chickens.

Let us assume that the chickens lay eggs in a synchronized way, or else our starting point and total the amount of time elapsed become ambiguous. Then, if we want exactly 14 eggs, the number of chickens we should use must divide 14. And, if this occurs for d days, the number of eggs e we will then have is given by

$$e = c\left\lfloor \frac{2}{3}d \right\rfloor ,$$

where $\lfloor x \rfloor$ is the greatest integer less than x. We use the floor function since we know that 1 chicken takes $\frac{3}{2}$ days to lay an egg, and in between these $\frac{3}{2}$ day benchmarks, there are no fractional eggs added. We have either given the chickens enough time to each lay a whole egg or not.

Since we need only such a small number of eggs (relatively speaking), we can just check the number of days we need with 14, 7, 2, and 1 chicken(s). We would find that the combinations that work are (14 chickens, 2 days), (7 chickens, 3 days), (7 chickens, 4 days), (2 chickens, 11 days), (1 chicken, 21 days), (1 chicken, 22 days), for a total of 6 ways. There are multiple solutions for some values of chickens because adding that extra day does not allow another full egg-laying period.

What if we consider situations where we do not have chickens laying eggs some of the time? For example, suppose I start with 3 chickens. I keep all of them for 4 egg laying periods, or 6 days. Then I have 12 eggs. I then keep 2 for one more full egg laying period, to get a total of 14.

This will become very complicated if we let it. For instance, I could have 2 chickens laying eggs for 2 days, giving 2 eggs. I then let these chickens go, and bring in 2 new chickens laying eggs for 2 days, giving 2 eggs. I then have 4 chickens laying 4 eggs in 4 days. But, if I had only let them lay eggs for $\frac{3}{2}$ days, I would have had 4 chickens laying 4 eggs in 3 days. Even worse, I could consider after this point having 0 chickens laying eggs for however many days I wished, or any number of chickens laying eggs, but only for periods less than $\frac{3}{2}$ days at a time. To prevent this, I will require that every chicken involved has to lay at least 1 egg, and the same chickens are used from day to day, and I can only release chickens (this makes differently ordered chicken day combinations non-unique).

The most chickens I could start with would be 14 chickens. After each chicken laid an egg, I would have exactly as much as I needed and would need to stop. Thus, I can write a solution (14,2), where the ordered pair is (chickens, days). Now, if I started with one fewer, I would have 13 eggs at the end of a day and a half, I would release 12, and the remaining one would lay the final egg after another day and a half. Thus, I have a solution $(12, \frac{3}{2}) + (1, 3)$. Note that, though I am working in intervals of $\frac{3}{2}$ days, the total number of days in this process is 3. This seems simple, but what about starting with 12 chickens? I will get $(10, \frac{3}{2}) + (2, 3)$ and also $(11, \frac{3}{2}) + (1, 5)$.

In general, we can see that the combinations needed will satisfy

$$\sum c_i \left\lfloor \frac{2}{3}d_i \right\rfloor = 14.$$

Suppose I know the function that tells me how many combinations there are to get exactly e eggs if I start with at most χ chickens, call this function $\phi(\chi, e)$. We know ϕ for two cases,

$$\phi(\chi, e) = \begin{cases} 0, & \text{for } \chi > e \\ 1, & \text{for } \chi = 1 \text{ or } \chi = e. \end{cases}$$

This is because there are no ways to get e eggs with χ chickens if I have more chickens than I need eggs, and each chicken has to lay one egg. Or, if I have exactly as many chickens as eggs and each chicken has to lay one egg, I will be done after just one egg-laying period and need to stop. Or, if I have one chicken, I need to keep it the whole time, and will never have a choice to release it at any iteration. We can also define $\phi(\chi, e)$ recursively,

$$\phi(\chi, e) = \sum_{i=1}^{\lfloor \frac{e}{\chi} \rfloor} \sum_{j=1}^{\chi-1} \phi(j, e - i\chi).$$

This function states that however many combinations starting with χ chickens and e eggs is the sum of the combinations of ways to get the remaining eggs if we free some number of chickens so that we have j, $j < \chi$ remaining after some whole number of egg laying periods. Notice that $\phi(\chi, e)$ is recursively defined in terms of smaller arguments, and it is known in the case where $\chi = 1$. Thus it can be solved for completely.

But, there is still another complication–this discusses the ways to get e eggs in terms of the time it takes for a chicken to lay one egg, not days. If I can wait extra days without the another full egg-laying period passing, I have more combinations. But, to do this, I'll need to consider the remainder of division of days elapsed by the egg-laying period. If this period is τ, then I can wait any extra number of days d such that d satisfies

$$\left\lfloor \frac{\lceil i\tau \rceil + d}{\tau} \right\rfloor = i,$$

where i is the number of periods I have already waited. This is to distinguish combinations such as starting with 3 chickens, releasing 2 after 3 days and keeping one an additional 2 days to get 7 eggs from starting with 3 chickens, releasing 2 after 4 days, and keeping one an additional day to get 7 eggs. Thus, I can distinguish (3 chickens, 3 days) + (1 chicken, 5 days) from (3 chickens, 4 days) + (1 chicken, 5 days). However, I will need to consider the total number of gestation periods n, since the indexes of my summations may change with respect to the days d I can add when the function recurs (note that, in this case, when I've waited an even number of gestation periods, I can add an extra day to wait to remove chickens, but if I've waited an odd number of them, I cannot, thus I need to keep track of the absolute index in these recursions). I can modify the sum as

$$\phi_\tau(\chi, e, n) = \sum_{i=1}^{\lfloor \frac{e}{\chi} \rfloor} \left(\left(\sum_{d=0}^{\lfloor \tau \rfloor} \left(1 + i + n - \left\lfloor \frac{\lceil (i+n)\tau \rceil + d}{\tau} \right\rfloor \right) \right) \sum_{j=1}^{\chi-1} \phi_\tau(j, e - i\chi, i + n) \right).$$

Our defining cases for ϕ still apply here; in the base cases, these are independent of reminder, and so we can recur downward to find ϕ_τ of any value.

Then the total number of ways to get e eggs is

$$\Phi_\tau(e) = \sum_{i=1}^{e} \phi_\tau(i, e, 0).$$

The reason I am not summing over all possible values of n and instead considering it to be 0 is because I have assumed that the chickens are at the beginning of their gestation periods when we start counting days.

Thus, if we need exactly 14 eggs and can release chickens as they lay them, the final answer is $\Phi_{\frac{3}{2}}(14)$ where Φ_τ and ϕ_τ are as defined above. Or, the answer is (14 chickens, 2 days), (7 chickens, 3 days), (7 chickens, 4 days), (2 chickens, 11 days), (1 chicken, 21 days), (1 chicken, 22 days) if we must have exactly 14 eggs, and do not release any chickens during the process (for a total of 6 combinations).