Problem of the Fortnight 7

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A fast mouse runs along the x-axis in the positive direction with speed m. A fat cat is sitting at the point $(0, C)$. When the mouse is at the origin the cat sees him and decides to chase. The cat runs with speed c, which is slower than the mouse. What path should the cat take so that he comes as close as possible to the mouse?

Solution. Consider a circle and a point. The shortest path from the point to the circle is the line going through that point and the center of the circle.

Now given that, consider a circle of radius ct, centered at $(0, C)$: that is the possible positions of the cat at time t. Then at time t, the closest possible distance between the two is given as:

$$f(t) = \sqrt{c^2 + (mt)^2} - ct$$

Now we need to find t so that the distance is minimized - then basic trigonometry would give us the direction towards which the cat should run. Now setting and solving $f'(t) = 0$, we would get

$$t = \frac{Cc}{m(m^2 - c^2)}$$

(it is obvious that $f''(t) < 0$, so this is a minimum). Then the angle the cat should have ran towards, from the y-axis to the directional vector, is

$$\arctan\left(\frac{mt}{C}\right) = \arctan\left(\frac{c}{m^2 - c^2}\right). \qquad \square$$