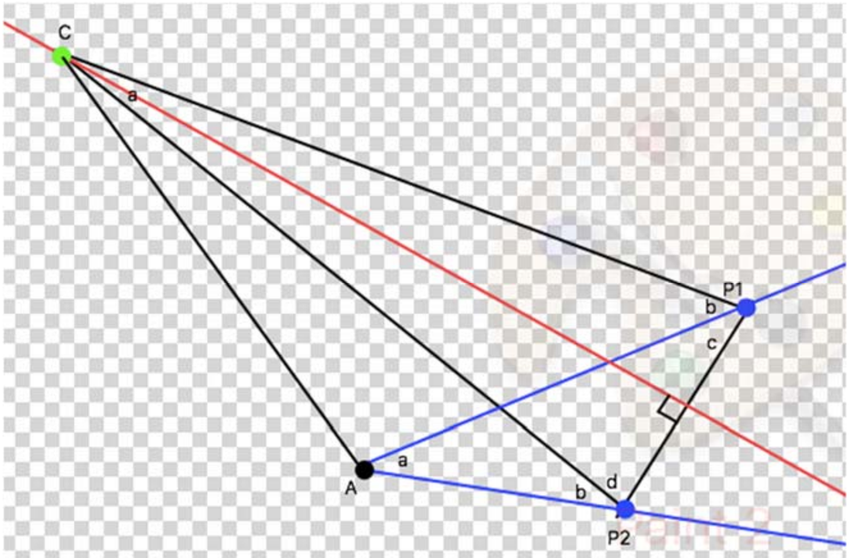
Kevin Sheridan, Jr.

Problem of the Fortnight #5



I start this problem by extending lines 1 and 2 (the blue lines) to a point, A, where they intersect forming angle P1AP2 or angle a. Then I draw a new line P1P2 and perpendicularly bisect it with another line, (the red line). The red line is the line of infinite points which are equal distance from P1 as they are from P2. On this line is where the solution will be. Point C, the solution, must form an angle CP1A equal to the angle CP2A, or angle b. The way to solve where point C is on the red line is to solve for angle b. To find angle b one additional line had to be drawn, AC. Labeling angle AP1P2, c, and angle AP2P1, d, I discovered that angle P1CP2 was equal to angle a. Assuming the lengths of lines AP1, P1P2, and P2A, can be measured, angle b can be found using the Law of Cosines:

(* see referee's note, next page *)

(CA)^2 = (AP1)^2 + (CP1)^2 - 2(AP1)(CP1)cos(b) and

(CA)^2 = (AP2)^2 + (CP2)^2 - 2(AP2)(CP2)cos(b) and (CP1) = (CP2) so

(AP1)^2 - 2(AP1)(CP1)cos(b) = (AP2)^2 - 2(AP2)(CP1)cos(b)

((AP1)^2 - (AP2)^2) / (2(CP1)(AP1-AP2) = cos(b)

arccos([(AP1)^2 - (AP2)^2] / [2(CP1)(AP1-AP2)]) = b

Note that

sin(a/2) = ((P1P2)/2) / (CP1) so

(CP1) = ((P1P2)/2) / (sin(a/2))

And that

[(AP1)^2 - (AP2)^2] / (AP1-AP2) = (AP1)+(AP2)

Together

b = arccos([(AP1)+(AP2)] (sin(a/2)) / (P1P2))

This equation finds the angle at which a line must be drawn from P1 or P2 to intersect the red line finding point C assuming you own a tool to measure angle a. If not then the following equation, found with more Law of Cosines can be used in terms of AP1, P1P2, and P2A

b=arccos([(AP1)+(AP2)][sin([arccos([(AP1)^2+(AP2)^2-(P1P2)^2]/[2(AP1)(AP2)]]/2)]/(P1P2))

I don't think I can simplify it.

(*) Referee's comment: Kevin locates C by using the angle b. Another way to locate C is by using the angle he has labeled d. Let B be the point where the red line intersects segment P_1P_2. Angle d is one angle of the right triangle BCP_2. The angles of the triangle add up to 180°, and so $d = 90° - a/2$. The angle d can be used (as Kevin does with angle b) to draw the line from P_2 to intersect the red line at C. – R. L. Foote