The Anh Pham

Solution:

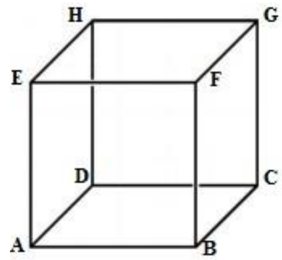
Suppose that we have an $n \times n$ square. It means that the square can be divided into $n \times n$ sub-squares.

For an ant to move between two opposite corners of this square along the edges of the sub-squares, it needs to move right and down. The number of right and down moves must be equal because of the square symmetry and each half of it has one right and one move edges.

Hence, the minimal length for the ant to move between two opposite corners is $2n$.

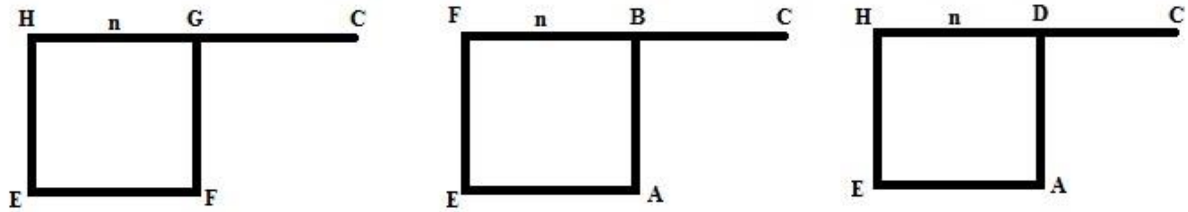
Let right move be R, and down move be D. The number of minimal paths along the edges of sub-squares now becomes the number of combinations that we can put nR and nD in a string of $2n$, which is C_n^{2n}.

Now suppose that we have a $n \times n \times n$ cube $ABCD.EFGH$. And ant wants to crawl between 2 opposite corners of the cube, so without losing the objectivity, suppose that we need to find the number of path with minimal length from C to E.



Imagine that $EGCA$ is a $2n \times n$ rectangular with FB in the middle. By a similar proof as above, we get C_n^{3n} paths with minimal lengths from C to E. Do it again with six rectangles combined from six pairs of the cube faces: $(EFBA, FGCB)$, $(EHGF, FGCB)$, $(EHGF, HGCD)$, $(EHDA, DABC)$, $(EHDA, HGCD)$, & $(EFBA, ABCD)$, we get the total number of minimal paths is $6C_n^{3n}$.

However, it is obvious that we must count many paths more than once, so we need to find how to subtract those from $6C_n^{3n}$. It is easy to see that when a cube is divided into smaller cubes, the paths that do not pass any entire cube edge from E to C, say EF, or BC can not be counted more than once. Only those include a cube edge get replicated in our counting. We can keep C to E fixed, and here I keep C. Any path that goes to C that includes CG, CB, or CD must be subtracted twice; because each of these edges is only included in three faces pairs. From C to E (as below), there are $3C_n^{2n}$ paths.



Therefore, the total shortest path without replication along the edges of the $n \times n$ cube is $6C_n^{3n} - 2 \times 3\,C_n^{2n}$. Applying this formula to the problem, we get $6C_4^{12} - 6C_4^{8} = 2550$ paths.