Problem of the Fortnight # 2

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Since we are withdrawing money at the end of the year, considering any form of interest that compounded more frequently than that would reduce to the annual compounding interest problem but for a different value of r (for convenience, the r considered here is the interest rate added to one).

So, considering the annually compounding case, with an initial principle P, when we start we will have

$$P$$

After one year, we will have

$$Pr - 1^2$$

After two,

$$Pr^2 - r(1^2) - 2^2$$

And after n years we will have

$$Pr^n - r^{n-1} - 2^2r^{n-2} - 3^2r^{n-3} - ...$$

$$Pr^n - \sum_{k=0}^{n} k^2r^{n-k}$$

Since this process should continue indefinitely, then the soonest we can run out of money is after an infinitely long time. So, for in the limit of large n, we have

$$Pr^n - \sum_{k=0}^{n} k^2r^{n-k} > 0$$

$$Pr^n > \sum_{k=0}^{n} k^2r^{n-k}$$

$$P > \sum_{k=0}^{n} k^2r^{-k}$$

Applying the limit,

$$P = \lim_{n\to\infty} \sum_{k=0}^{n} k^2r^{-k}$$

$$P = \sum_{k=0}^{\infty} k^2r^{-k}$$

If we think of r as a variable instead of a constant, then

$$f(r) = \sum_{k=0}^{\infty} k^2r^{-k}$$

or better yet,

$$f(\frac{1}{x}) = \sum_{k=0}^{\infty} k^2x^k$$

Here, $f(\frac{1}{x})$ is really a placeholder, and we will not manipulate it directly. Instead, we can rearrange the right hand side to show

$$f(\frac{1}{x}) = x\sum_{k=0}^{\infty} k(kx^{k-1})$$

where I have factored out an x from each term in the sum and broke up the k squared to make a specific term more recognizable, kx^{k-1} or $\frac{d}{dx}r^x$. Then

$$f(\frac{1}{x}) = x\sum_{k=0}^{\infty} k(kx^{k-1}) = x\sum_{k=0}^{\infty} x\frac{d}{dx}x^k = x(\frac{d}{dx}\sum_{k=0}^{\infty} kx^k)$$

since the sum of the derivatives in the derivative of the sum. The same trick can be used to evaluate the remaining sum,

$$\sum_{k=0}^{\infty} kx^k = x\sum_{k=0}^{\infty} kx^{k-1} = x\sum_{k=0}^{\infty} \frac{d}{dx}x^k = x\frac{d}{dx}\sum_{0}^{\infty} x^k$$

But this last sum is a geometric series with common ratio x, and its value is known for $|x| < 1$, which is $\frac{1}{1-x}$. The complete expression now becomes

$$f(\frac{1}{x}) = x(\frac{d}{dx}(x\frac{d}{dx}(\frac{1}{1-x})))$$

$$f(\frac{1}{x}) = x(\frac{d}{dx}(\frac{x}{(1-x)^2}))$$

$$f(\frac{1}{x}) = x(\frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}) = \frac{x(1+x)}{(1-x)^3}$$

Since we required that $|x| < 1$, r, the reciprocal of x, the actual parameter we are plugging into the function, must satisfy $|r| > 1$. This works, since r was the interest rate added to one.

Then evaluating this for $x = \frac{1}{r}$,

$$f(r) = \frac{\frac{1}{r}(1+\frac{1}{r})}{(1-\frac{1}{r})^3} = \frac{r^3(\frac{1}{r})(1+\frac{1}{r})}{r^3(1-\frac{1}{r})^3} = \frac{r(r+1)}{(r-1)^3}$$

Because the interest was assumed to be compounding annually with a rate of 0.05, r is 1.05. Plugging this in to the above equation gives that \$17,220.00 is the smallest initial principle P for which this can be done.