Math. Z. 194, 331–350 (1987)

Mathematische Zeitschrift

Differential Geometry of Real Monge-Ampère Foliations

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Introduction

The foliations studied in this paper are those that arise from the Monge-Ampère condition requiring the Hessian of a function defined on an affine manifold to have rank that is one less than maximal.

The primary motivation for this study comes from work due to Stoll, Burns, and Bedford, who have studied Monge-Ampère foliations of complex manifolds (see [15, 3, 1]).

Much of this paper originally appeared as part of the author's doctoral dissertation, [5] which was completed at the University of Michigan under the direction of Daniel M. Burns. The author would like to express thanks for the many enlightening discussions with him.

A brief outline of the paper follows, as well as a discussion of applications to the complex case.

Let $\Omega \subset \mathbb{R}^n$ be open and connected, and let u: $\Omega \to \mathbb{R}$ be sufficiently differentiable. (The reader should note that in the last two sections Ω is taken to be a subset of $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$ for notational purposes. In this introduction it may be assumed, except where indicated otherwise, that all functions, surfaces, vector fields, and forms are C^4. The exact regularity needed is addressed in the main body of the paper.) Let $H(u)$ denote the Hessian of u. The real, homogeneous, Monge-Ampère equation is

$$(*) \qquad \det H(u) = 0.$$

The special cases of $(*)$ of interest are those of constant rank, especially $\operatorname{rk} H(u) = n-1$.

Theorem A. *Let u satisfy $\operatorname{rk} H(u) = n-k$ for some $1 \leqq k \leqq n$. Then Ω is foliated by k-planes in such a way that u is linear on each leaf of the foliation.*

The standard solution of $(*)$ is the function $u(x) = (\sum x_k^2)^{1/2}$ defined on $\Omega = \mathbb{R}^n - \{0\}$. Then main properties of the standard solution are a) $\operatorname{rk} H(u) = n$

-1, b) the leaves of the foliation are transverse to the level sets of u, and c) u is convex. For any solution of (*) satisfying these three properties a canonical vector field $\frac{\partial}{\partial u}$ is introduced. The covariant derivative $S=\tilde{\nabla}\frac{\partial}{\partial u}$ is a measure of the "twist" of the foliation, where $\tilde{\nabla}$ is the standard connection on $\mathbb{R}^n$. The relationships between $H(u)$, $\frac{\partial}{\partial u}$, S, and the geometry of the foliation are studied. For example:

Theorem B. *$\nabla S=0$ on Ω if and only if the leaves of the foliation are parallel or meet in a single point when extended. (The operator ∇ is the connection obtained by restricting $\tilde{\nabla}$ to the level sets of u.)*

Note that the foliation of the standard solution has the property in this theorem, since the leaves are the lines through the origin in $\mathbb{R}^n$.

The two main theorems of the second section (17 and 18) are characterizations of the standard solution.

Theorem C. *Let u satisfy conditions* a)–c) *above. Suppose that some level set of u has a compact component, and that $\nabla H(u)=0$ on Ω. Then (after appropriate affine changes) u is the restriction to Ω of the standard solution.*

Theorem D. *Let Ω contain 0 and be star-shaped with respect to 0. Let $u\colon \Omega\to\mathbb{R}$ be continuous with $u(0)=0$, $u(x)>0$ for $x\neq 0$, and a solution of (*) on $\Omega-\{0\}$. Suppose that $\tau=f(u)$ is strictly convex on a neighborhood V of 0 where $f\colon \mathbb{R}\to\mathbb{R}$. Then (after an affine change of coordinates) u is the restriction to Ω of the standard solution.*

As an application, the following theorem (due to Burns [3]) is proved. It is the real counterpart of Stoll's characterization of balls in $\mathbb{C}^n$. (See Theorem H below.)

Theorem E. *Let M be a connected n-dimensional affine manifold. Let $\tau\colon M\to[0,R^2)$, $0<R\leqq\infty$, be proper and strictly convex. Suppose that $u=\sqrt{\tau}$ satisfies (*) on $M-\tau^{-1}(0)$. Then there is an affine diffeomorphism $F\colon B_R(0)\to M$, where $B_R(0)\subset\mathbb{R}^n$ is the ball of radius R, such that $u\circ F$ is the standard solution of (*).*

In the final section Cauchy problems for (*) are investigated. A feature of the existence parts of the following theorems is their constructive nature.

Theorem F. *Let $M\subset\mathbb{R}^n$ be a hypersurface, let $\phi,\psi\colon M\to\mathbb{R}$ and let N be a unit normal vector along M. If $H_M(\phi)+\psi H(r)$ is definite on TM, then there exists a unique function u, defined on a neighborhood Ω of M, that satisfies (*) on Ω and $u=\phi$, $Nu=\psi$ on M. Here $H_M(\phi)$ denotes the Hessian of ϕ relative to M, and $r\colon \Omega\to\mathbb{R}$ is the signed distance function for M.*

Theorem G. *Let $M\subset\mathbb{R}^n$ be a hypersurface with non-vanishing Gaussian curvature. Let ω be a one-form along M with $\omega=0$ on TM and $\omega(X)\neq 0$ for vectors X transverse to M. Then there exists a unique function u, defined on a neighborhood Ω of M, that satisfies conditions* a)–c) *on Ω and $u=0$, $du=\omega$ on M.*

Although the subject of this paper belongs to the realm of affine differential geometry, the relationship between $(*)$ and its complex analog should be discussed. In particular, some assumptions about the nature of the functions involved reduce the complex case to the real case, and the techniques and results of the present paper apply.

Let Ω be a complex manifold of dimension n. Let $u\colon \Omega \to \mathbb{R}$. The complex, homogeneous, Monge-Ampère equation is

$$(**) \qquad (\partial\bar{\partial}u)^n = 0,$$

where $\partial\bar{\partial}u$ is the Levi form of u. If $\mathbb{C}H(u)$ denotes the complex Hessian matrix of u with respect to a local holomorphic coordinate system, then $(**)$ is equivalent to

$$(**)' \qquad \det \mathbb{C}H(u) = 0.$$

As in the real case, if $\operatorname{rk}\partial\bar{\partial}u = n-k$ on Ω, then Ω is foliated by k-dimensional complex submanifolds in such a way that u is pluriharmonic on each leaf. The standard example is $u(z) = \log|z|^2$ on $\Omega = \mathbb{C}^n - \{0\}$. The leaves are the complex lines through the origin. (See [2, 3, 15] for details.)

The following is Stoll's characterization of balls in $\mathbb{C}^n$ in terms of solutions of $(**)$.

Theorem H. (Stoll [15], Burns [3]) *Let M be a connected, complex manifold of dimension n. Let $\tau\colon M \to [0, R^2)$, $0 < R \leqq \infty$, be C^∞, proper, and strictly plurisubharmonic (spsh). Suppose that $u = \log\tau$ satisfies $(**)$ on $M - \tau^{-1}(0)$. (Stoll has called a pair (M, τ) with these properties a strictly parabolic manifold.) Then there is a biholomorphism $F\colon B_R(0) \to M$, where $B_R(0) \subset \mathbb{C}^n$ is the ball of radius R, such that $u \circ F$ is the standard solution of $(**)$.*

It should be noted that $\tau \in C^\infty$ is necessary for the proofs in [15] and [3] when $R < \infty$. This is in contrast to the real case (Theorem E) where one only needs $\tau \in C^4$. Burns gives an alternate proof of Theorem H for $R = \infty$ using only $\tau \in C^5$. It is not known if $\tau \in C^5$ suffices in general for $R < \infty$. (See Theorem J below.)

The function $u\colon \Omega \to \mathbb{R}$ is *locally Reinhardt* if Ω can be covered with coordinate charts on which u is a Reinhardt function. If $\zeta = (\zeta_1, \ldots, \zeta_n)$ is such a coordinate system, let $z_k = \log\zeta_k$ locally. Then for the coordinates $z = x + iy$ we have that u is independent of $y = (y_1, \ldots, y_n)$, and $(**)'$ becomes $(*)$. With this reduction from the complex case to the real case the following theorems can be proved, the details of which appear in subsequent submissions of the author [8, 9]. The first one is a partial solution of a conjecture due to Burns [4, 7].

Theorem I. *Let M be a complex manifold of dimension n. Let $\tau\colon M \to [0, \infty)$ be spsh and C^∞. Suppose there is a compact set $K \subset M$ such that on $\Omega = M - K$ the function $u = \log\tau$ satisfies $(**)$. (A pair (M, τ) with these properties is called strictly parabolic at infinity.) If additionally u is locally Reinhardt on Ω and strictly exterior (a technical generic assumption), then M can be biholomorphically identified with an algebraic submanifold of $\mathbb{C}^N$.*

Theorem J. *Theorem H holds if "$\tau \in C^\infty$" is replaced by "$\tau \in C^5$ and u is locally Reinhardt on $M - \tau^{-1}(0)$".*

General Real Monge-Ampère Foliations

Let $\Omega \subset \mathbb{R}^n$ be open, and let $u: \Omega \to \mathbb{R}$ be C^2. The *Hessian* of u is the symmetric tensor $H(u) = \tilde{\nabla} du$ where $\tilde{\nabla}$ is the standard flat connection on $\mathbb{R}^n$. If X and Y are vectors at a point $p \in \Omega$, then $H(u)(X, Y) = (\tilde{\nabla}_X du)(Y) = X(du(Y)) - du(\tilde{\nabla}_X Y)$, where Y is extended in a smooth way to a neighborhood of p. This definition makes sense on an arbitrary manifold with a torsion-free connection.

Much of the material in this paper is local in nature, and could take place on an arbitrary *affine manifold*, that is, a manifold with a torsion-free connection $\tilde{\nabla}$ whose curvature tensor $\tilde{R}(X, Y) = \tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}$ vanishes identically. The absence of curvature will play an important role in many proofs and constructions. If $(x_1, \ldots, x_n)$ is an affine coordinate system on an affine manifold, then

$$H(u) = \sum \frac{\partial^2 u}{\partial x_i \partial x_j} dx_i \otimes dx_j.$$

The solutions of the real Monge-Ampère equation $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = 0$ that give rise to well-behaved foliations are those of constant rank:

$$\operatorname{rk} H(u) = \operatorname{rk}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = n - k.$$

Theorem 1. *Let $u: \Omega \to \mathbb{R}$ be C^3 with $\operatorname{rk} H(u) = n - k$. Then there is a foliation of Ω by k-planes, such that u is linear on each leaf of the foliation.*

Proof. This is well-known (see [11]), but a simple invariant proof is given here.

Let

$$\begin{aligned} \mathscr{F} = \ker H(u) &= \{X \in T\Omega \mid H(u)(X, Y) = 0 \text{ for all } Y \in T\Omega\} \\ &= \{X \in T\Omega \mid \tilde{\nabla}_X du = 0\}. \end{aligned}$$

This is a C^1 k-distribution (in the sense of differential topology). Let X and Y be C^1 sections of $\mathscr{F}$. Let Z be an arbitrary C^1 vector field. Then

$$\begin{aligned} H(u)(Z, \tilde{\nabla}_X Y) &= (\tilde{\nabla}_Z du)(\tilde{\nabla}_X Y) = X((\tilde{\nabla}_Z du)(Y)) - (\tilde{\nabla}_X \tilde{\nabla}_Z du)(Y) \\ &= -(\tilde{\nabla}_Z \tilde{\nabla}_X du)(Y) - (\tilde{\nabla}_{[X,Z]} du)(Y) = 0, \end{aligned}$$

since $X, Y \in \mathscr{F}$ and there is no curvature. Similarly, $H(u)(Z, \tilde{\nabla}_Y X) = 0$. It follows that $\tilde{\nabla}_X Y$, $\tilde{\nabla}_Y X$, and $[X, Y]$ are sections of $\mathscr{F}$. Thus $\mathscr{F}$ is integrable and the leaves of the resulting foliation are totally geodesic (see [14], p. 32).

Finally, in directions tangent to the leaves, $H(u) \equiv 0$, and so u is linear (du is parallel) on each leaf. Q.E.D.

Clearly the domain of u can be made larger by extending the planes of the foliation beyond Ω and by letting u continue to be linear on the planes. This will be well-defined and will satisfy $\operatorname{rk} H(u)=n-k$ until the planes start to interest each other. A discussion of this for one-dimensional foliations is taken up in the next section.

One-Dimensional Foliations

The *standard solution* of $\det H(u)=0$ is the function $u_s(x)=r(x)=|x|=(\sum x_k^2)^{\frac{1}{2}}$ defined on $\Omega=\mathbb{R}^{n+1}-\{0\}$ (note the change in dimension). It is easily seen that this is a solution by observing that u_s is convex, and so $H(u)\geqq 0$, and that $H(u_s)\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)=\frac{\partial^2}{\partial r^2} r=0$, where $\frac{\partial}{\partial r}$ is the radial field. It follows that $H(u_s)\left(\frac{\partial}{\partial r}, X\right)=0$ for all vectors X, and so $\operatorname{rk} H(u_s)<n+1$. The level sets of u_s are strictly convex, thus $\operatorname{rk} H(u_s)\equiv n$. The leaves of the resulting foliation are the lines through the origin.

The properties of the standard solution of interest in the remainder of this section are

(1) a) $\operatorname{rk} H(u)=n$ (one less than maximal),
b) the leaves of the foliation are transverse to the level sets of u, and
c) u is convex, that is, $H(u)(X, X)\geqq 0$ for all vectors X.

It will be assumed in the rest of this section, unless otherwise indicated, that $u\colon \Omega\to\mathbb{R}$ is a C^4 function that satisfies (1), where $\Omega\subset\mathbb{R}^{n+1}$ is open and connected. Theorems 17 and 18 characterize u_s out of all such functions. Note that the conditions in (1) imply that the level sets of u are strictly convex, and that $du(X)\neq 0$ if $X\neq 0$ is tangent to the foliation.

Property b) should be thought of as a non-degeneracy condition. Property c) will allow the tensor that measures the "twist" of the foliation to be diagonalized.

When using local coordinates or a local frame, the indices $0, 1, \ldots, n$ will be used. The index 0 will be reserved for the leaf direction.

Definition 2. Let $\frac{\partial}{\partial u}$ be the unique vector field tangent to the foliation for which $du\left(\frac{\partial}{\partial u}\right)\equiv 1$.

The field $\frac{\partial}{\partial u}$ satisfies $H(u)\left(\frac{\partial}{\partial u}, \cdot\right)=\tilde{V}_{\partial/\partial u}\, du=0$. This implies that the tangent spaces of the level sets of u are parallel at all points on a given leaf. The integral curves of $\frac{\partial}{\partial u}$ are just the straight lines of the foliation. Since the leaves are geodesics, the field $\tilde{V}_{\partial/\partial u}\frac{\partial}{\partial u}$ is tangent to the leaves. But

$$du\left(\tilde{V}_{\partial/\partial u}\frac{\partial}{\partial u}\right)=\frac{\partial}{\partial u}\left(du\left(\frac{\partial}{\partial u}\right)\right)-(\tilde{V}_{\partial/\partial u}\, du)\left(\frac{\partial}{\partial u}\right)=0,$$

so $\tilde{V}_{\partial/\partial u}\frac{\partial}{\partial u}=0$, and the flow of $\frac{\partial}{\partial u}$ is geodesic.

Definition 3. Let Φ_r be the flow of $\frac{\partial}{\partial u}$, that is, $\Phi\colon \Omega\times\mathbb{R}\to\mathbb{R}^{n+1}$,

$$\Phi_r(p)=\Phi(p,r)=p+r\left.\frac{\partial}{\partial u}\right|_p.$$

For $a\in\mathbb{R}$, let $\Sigma_a=\{p\in\Omega\,|\,u(p)=a\}$. Notice that $\Phi_r\colon \Sigma_a\to\Sigma_{a+r}$ is a diffeomorphism provided that $\Phi\colon \Sigma_a\times[0,r]\to\mathbb{R}^{n+1}$ is a diffeomorphism onto its image. This is the case for all sufficiently small r when the level sets are compact. The map Φ is clearly the device needed to extend the definition of u outside Ω. The value of u at $q=\Phi_r(p)=p+r\left.\frac{\partial}{\partial u}\right|_p$ should be $u(p)+r$. For this definition to make sense, q can lie on only one line of the foliation (when extended), and Φ_r must be a diffeomorphism from some neighborhood of p to some neighborhood q. More will be said later about extending the domain of u.

Definition 4. Let $S=\tilde{\nabla}\frac{\partial}{\partial u}$, that is, for $X\in T\Omega$, let $SX=\tilde{\nabla}_X\frac{\partial}{\partial u}$.

This tensor is a measure of the "twist" of the foliation. (Compare this with the "twist" tensor of Bedford and Burns in [1] and [3].) In particular, $S\equiv 0$ if and only if the lines of the foliation are parallel. In this case the domain of u can be extended indefinitely by following the lines of the foliation in both directions, and the graph of u is a cylinder over any of its level sets.

Clearly, $S\frac{\partial}{\partial u}=0$. If X is tangent to a level set Σ, then

$$du(SX)=du\left(\tilde{\nabla}_X\frac{\partial}{\partial u}\right)=X\left(du\left(\frac{\partial}{\partial u}\right)\right)-(\tilde{\nabla}_X du)\left(\frac{\partial}{\partial u}\right)=0.$$

Thus $SX\in T\Sigma$, and S will usually be viewed as a map $S\colon T\Sigma\to T\Sigma$.

Example 5. a) Let $\Sigma\subset\mathbb{R}^{n+1}$ be a strictly convex hypersurface, and let $p\in\mathbb{R}^{n+1}-\Sigma$. Define $u(p)=0$, $u(\Sigma)=1$. For $q\in\Sigma$ and $r\in(0,\infty)$, let $u((1-r)p+rq)=r$. This is well-defined if the map $\Sigma\times(0,\infty)\to\mathbb{R}^{n+1}$, $(q,r)\to(1-r)p+rq$ is a diffeomorphism onto its image. Then u or $-u$ satisfies (1). u is convex if p is "inside" Σ, concave if p is "outside" Σ. Along the ray $r\to(1-r)p+rq$ the vector $\frac{\partial}{\partial u}$ is $q-p$ or $p-q$. Clearly any two different values can be chosen for $u(p)$ and $u(\Sigma)$. For solutions of this type, it is easy to show that $S=\frac{1}{u-u(p)}I\colon T\Sigma\to T\Sigma$. When p is the origin and Σ is the unit n-sphere, then u is the *standard solution*, $u_s(x)=r(x)=|x|=(\sum x_k^2)^{\frac{1}{2}}$, and $\frac{\partial}{\partial u}$ is the radial field $\frac{\partial}{\partial r}$ on $\mathbb{R}^{n+1}$.

b) Let $\Sigma\subset\mathbb{R}^{n+1}$ be a strictly convex hypersurface, and let N be a smooth vector field along Σ that is transverse to Σ, "outward-pointing", and that satisfies $\tilde{\nabla}_X N\in T\Sigma$ for any $X\in T\Sigma$. The map $\Sigma\times\mathbb{R}\to\mathbb{R}^{n+1}$, $(p,r)\to p+rN_p$ is a diffeomorphism onto some neighborhood of Σ. On that neighborhood define $u(p+rN_p)=r$. Then u satisfies (1). The field $\frac{\partial}{\partial u}$ is obtained by parallel-translating N along the lines $r\to p+rN_p$.

c) See Corollary 21 below.

Proposition 6. a) *In Example 5a, if Σ is C^k, $k \geqq 2$, then u is C^k.*
b) *In Example 5b, if Σ is C^k and N is C^{k-1}, $k \geqq 2$, then u is C^k.*

Proof. a) This is clear. b) From $u(p+rN_p)=r$, it is seen that u is C^{k-1}, hence differentiable. The foliation is C^{k-1}, and so the decomposition $X=Y+fN$, $Y \in T\Sigma$, is C^{k-1}, that is, if X is a smooth vector field, then the vector field Y and the function f are C^{k-1}. From $Xu=du(X)=f$, it follows that du is C^{k-1}. Q.E.D.

The quantities $\frac{\partial}{\partial u}$, S, and $H(u)$ are similar to the unit normal N, the Weingarten map W, and the second fundamental form II of a hypersurface in a Riemannian manifold. (See, e.g., [12].) In fact, if $\Sigma \subset \mathbb{R}^{n+1}$ is strictly convex with outward unit normal N, then the construction of Example 5b yields the distance function

$$u(p)=\begin{cases} \operatorname{dist}(p,\Sigma) & \text{for } p \text{ outside } \Sigma \\ -\operatorname{dist}(p,\Sigma) & \text{for } p \text{ inside } \Sigma. \end{cases}$$

Along Σ one has $\frac{\partial}{\partial u}=N$, $S=W$, and $H(u)=\mathrm{II}$. In this context, note that Proposition 6b gives a proof that the distance function for a C^k hypersurface is C^k when $k \geqq 2$. (For a more direct proof see [6].)

Lemma 7. $\left(\text{Basic properties of } S, \Phi, \frac{\partial}{\partial u}, \text{ and } H(u)\right)$

a) *S is symmetric with respect to $H(u)$.*
b) $(\Phi_r)_*(X)=X+rSX$.
c) $S((\Phi_r)_*(X))=SX$.
d) *If $SX=kX$, then $S((\Phi_r)_*(X))=\frac{k}{kr+1}(\Phi_r)_*(X)$ as long as $kr+1 \neq 0$.*
e) $\tilde{\nabla}_{\partial/\partial u} S=-S^2$.
f) $(\tilde{\nabla}_{\partial/\partial u} H(u))(X,Y)=-H(u)(SX,Y)$.
g) $(\tilde{\nabla}_X S)Y=(\tilde{\nabla}_Y S)X$.
h) *$(\tilde{\nabla}_X H(u))(Y,Z)$ is symmetric in X, Y, Z.*

Remark. In b) and c), and wherever else it is necessary, the following convention will be used. If p and q are on the same leaf, then $T_p\Sigma_{u(p)}$ and $T_q\Sigma_{u(q)}$ are identified by affine parallel translation along the leaf. Care must be exercised in this identification, however, since S varies from one level set to another.

Proof. a)
$$\begin{aligned} H(u)(SX,Y)-H(u)(X,SY) &=(\tilde{\nabla}_Y du)(SX)-(\tilde{\nabla}_X du)(SY) \\ &=Y(du(SX))-du(\tilde{\nabla}_Y SX)-X(du(SY))+du(\tilde{\nabla}_X SY) \\ &=-du\left(\tilde{\nabla}_Y \tilde{\nabla}_X \frac{\partial}{\partial u}\right)+du\left(\tilde{\nabla}_X \tilde{\nabla}_Y \frac{\partial}{\partial u}\right) \\ &=du(S[X,Y])=0. \end{aligned}$$

b) This follows by differentiating $\Phi_r(p)=p+r\left.\frac{\partial}{\partial u}\right|_p$ in the direction $X\in T_p\Omega$. Here SX is computed at p, and the result is translated to $\Phi_r(p)$.

c) Let $\sigma(t)$ be a curve fitting $X\in T_p\Omega$, that is, $\sigma(0)=p$, $\sigma'(0)=X$. Then $\Phi_r\circ\sigma$ fits $(\Phi_r)_*(X)$. It follows that

$$SX=\frac{d}{dt}\left(\left.\frac{\partial}{\partial u}\right|_{\sigma(t)}\right)_{t=0} \quad \text{and} \quad S((\Phi_r)_* X)=\frac{d}{dt}\left(\left.\frac{\partial}{\partial u}\right|_{(\Phi_r\circ\sigma)(t)}\right)_{t=0}.$$

These are equal since $\left.\frac{\partial}{\partial u}\right|_p=\left.\frac{\partial}{\partial u}\right|_{\Phi_r(p)}$.

d) This follows from b).

e) Let $X_p\in T_p\Omega$, and extend it to a field X along the leaf through p by $X_{\Phi_r(p)}=(\Phi_r)_*(X_p)$. By c), the field SX is parallel along the leaf. Then

$$\begin{aligned}(\tilde{V}_{\partial/\partial u}S)(X_p)&=(\tilde{V}_{\partial/\partial u}(SX)-S(\tilde{V}_{\partial/\partial u}X))_p\\&=-S(\tilde{V}_{\partial/\partial u}(X_p+r\,SX_p))_p\\&=-S^2X_p,\end{aligned}$$

since $r=u-u(p)$ along the leaf.

f) Let $X, Y\in T_p\Omega$, and extend them along the leaf through p by parallel translation. Then

$$\begin{aligned}(\tilde{V}_{\partial/\partial u}H(u))(X,Y)&=\frac{\partial}{\partial u}(H(u)(X,Y))=\frac{\partial}{\partial u}((\tilde{V}_X du)(Y))\\&=(\tilde{V}_{\partial/\partial u}\tilde{V}_X du)(Y)=(\tilde{V}_{[\partial/\partial u, X]}du)(Y)=-H(u)\left(\tilde{V}_X\frac{\partial}{\partial u}, Y\right)\\&=-H(u)(SX,Y).\end{aligned}$$

g) Let X and Y be parallel fields. Then

$$\begin{aligned}(\tilde{V}_X S)Y-(\tilde{V}_Y S)X&=\tilde{V}_X(SY)-\tilde{V}_Y(SX)\\&=\tilde{V}_X\tilde{V}_Y\frac{\partial}{\partial u}-\tilde{V}_Y\tilde{V}_X\frac{\partial}{\partial u}=\tilde{V}_{[X,Y]}\frac{\partial}{\partial u}=0.\end{aligned}$$

h) This is true if u is any C^3 function on an affine manifold. Q.E.D.

Since $H(u)$ is positive definite on $T\Sigma$ for each level set Σ, the map S: $T\Sigma\to T\Sigma$ has eigenvalues $\lambda_1\leqq\ldots\leqq\lambda_n$, which are continuous on Ω.

Corollary 8. *Let $X_1,\ldots,X_n$ be an $H(u)$-orthogonal basis for $T_p\Sigma_{u(p)}$ consisting of eigenvectors of S. Extend this basis along the leaf through p by parallel translation. Then*

a) $X_1,\ldots,X_n$ is an $H(u)$-orthogonal basis of eigenvectors for $T_q\Sigma_{u(q)}$ for every q on the leaf.

b) The eigenvalues of S are given by $\lambda_k=\frac{a_k}{a_k u+b_k}$, $k=1,\ldots,n$, where a_k and b_k depend continuously on the leaves. If the level set Σ_0 meets all the leaves, then one can take $b_k=1$, in which case the $a_1,\ldots,a_n$ are the eigenvalues on Σ_0.

c) If $p \in \Sigma_0$ and $X_1, \dots, X_n$ are $H(u)$ - orthonormal at p, then the matrix for $H(u)$ in the frame $\frac{\partial}{\partial u}, X_1, \dots, X_n$ is $\operatorname{diag}\left(0, \frac{1}{a_1 u+1}, \dots, \frac{1}{a_n u+1}\right)$ along the leaf through p.

It follows from Corollary 8 that if Σ is a level set that meets every leaf of the foliation, then the eigenvalues of S on Σ determine the eigenvalues on every other level set.

The eigenvalues of S now have an obvious interpretation. If $SX = kX \neq 0$ for $X \in T_p \Sigma_{u(p)}$, then $(\Phi_{-1/k})_*(X) = X - \frac{1}{k} SX = 0$, and so $\Phi_{-1/k}$ is not a diffeomorphism in a neighborhood of p. Thus the level surface $\left\{u = u(p) - \frac{1}{k}\right\}$ does not meet the leaf through p. Intuitively, the leaves of the foliation passing through points near p (at least points along curves in $\Sigma_{u(p)}$ fitting X) intersect infinitesimally at $\Phi_{-1/k}(p)$. The solution u, when extended along the leaves, cannot be extended beyond $\Phi_{-1/k}(p)$. The surfaces $\{\Phi_{-1/\lambda_k}(p) \mid p \in \Sigma_a\}$ for $\lambda_k \neq 0$ are similar to the surfaces of focal points for a hypersurface in Euclidean space (see [13], §6).

If $\lambda_1 < 0 < \lambda_n$ at p, then the leaf through p is a portion of the line segment between $p - \frac{1}{\lambda_1} \frac{\partial}{\partial u}\Big|_p$ and $p - \frac{1}{\lambda_n} \frac{\partial}{\partial u}\Big|_p$. If the leaf is not this entire segment, the solution can be extended so that it will be, since the map $\Phi\colon \Sigma_{u(p)} \times \left(-\frac{1}{\lambda_n}, -\frac{1}{\lambda_1}\right) \to \mathbb{R}^{n+1}$ is a diffeomorphism in some neighborhood of $\{p\} \times \left(-\frac{1}{\lambda_n}, -\frac{1}{\lambda_1}\right)$. Similarly, if the eigenvalues at p are all non-negative or non-positive (resp. all zero), the solution can be extended so that the leaf through p is a ray (resp. a line). If this extension is made for all $p \in \Omega$, some multiple-valuedness may occur and the leaves will form a foliation only in the sense that "nearby" leaves will not intersect each other.

Everything done up to this point makes sense on an arbitrary affine manifold M of real dimension $n+1$, with some modifications. In particular, $\Phi_r(p) = p + r \frac{\partial}{\partial u}\Big|_p$ should be replaced by $\Phi_r(p) = \operatorname{Exp}_p\left(r \frac{\partial}{\partial u}\right)$ where $\operatorname{Exp}_p\colon T_p M \to M$ is the exponential map determined by the connection $\tilde{\nabla}$ on M. A hypersurface $\Sigma \subset M$ is strictly convex if $H(f)\colon T\Sigma \otimes T\Sigma \to \mathbb{R}$ is positive or negative definite for every local C^2 defining function f.

If $u\colon M \to \mathbb{R}$ is a solution of (1), then u can be extended to $u\colon M' \to \mathbb{R}$, $M \subset M'$, so that the leaves all have maximum possible length. The construction of M' is intuitively clear. Extend each leaf to the greatest length allowed by the eigenvalues. Then let M' be the disjoint union of these leaves with the obvious affine structure, and extend u linearly along each leaf. Note that this construction avoids the problem of multiple-valuedness mentioned above.

Definition 9. The solution u is *exterior* if u takes on arbitrarily large positive values on each leaf of the foliation.

Corollary 10. *The solution u is (or can be extended to be) exterior if and only if the eigenvalues of S are non-negative.*

Definition 11. An exterior solution is *strictly exterior* if the eigenvalues of S are positive.

We now return to the assumption that u: $\Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^{n+1}$ a domain. The reader is invited to determine which portions of the rest of the paper can be phrased in terms of an arbitrary affine manifold. Some results, e.g., Theorem 18, are local in nature and need little or no modification. Others, e.g., Theorem 15, require that a certain family of lines intersect in a single point – an event that doesn't necessarily happen in a general affine manifold.

The next two propositions should be compared with [12], p. 36.

Proposition 12. *Let Σ be a connected component of a level set of u. Then $S=kI$: $T\Sigma \to T\Sigma$, $k \neq 0$ a constant, if and only if the leaves of the foliation that pass through Σ meet, when extended, at a single point $p \in \mathbb{R}^{n+1}$.*

Proof. Suppose $S=kI$, and consider the map $\Phi_{-1/k}$: $\Sigma \to \mathbb{R}^{n+1}$. If $X \in T\Sigma$, then $(\Phi_{-1/k})_*(X)=0$ by Lemma 7b. Hence $\Phi_{-1/k}$ is a constant map. Let p be its image. Since $\Phi_r(q)$ is on the same leaf (or its extension) as q, it follows that p is on the extension of every leaf through Σ.

Conversely, suppose the leaves through Σ all meet at p. For $q \in \Sigma$ let $N_q = q - p$. Then $\frac{\partial}{\partial u} = fN$ for some function f: $\Sigma \to \mathbb{R}$. Note that f is C^2 since Σ is C^4 and $\frac{\partial}{\partial u}$ is C^2. For $X \in T_q\Sigma$, we have that $SX = \tilde{V}_X(fN) = (Xf)N + fX$. This is tangent to Σ, so f must be a constant. Q.E.D.

The sign of k determines whether p will be "inside" or "outside" Σ. Along the leaves passing through Σ the solution is of the type in Example 5a. If u is extended up to p, it can be made continuous at p by setting $u(p) = u(\Sigma) - \frac{1}{k}$.

Proposition 13. *Let Σ be as in Proposition 12, and suppose $n \geqq 2$. If $S=fI$: $T\Sigma \to T\Sigma$ where f: $\Sigma \to \mathbb{R}$ is C^1, then f is constant.*

Proof. Let X, $Y \in T\Sigma$ be linearly independent fields on an open set in Σ. Then

$$\begin{aligned}
0 &= \tilde{R}(X, Y)\frac{\partial}{\partial u} \\
&= \tilde{V}_X(SY) - \tilde{V}_Y(SX) - S[X, Y] \\
&= (Xf)Y + f\tilde{V}_X Y - (Yf)X - f\tilde{V}_Y X - f[X, Y] \\
&= (Xf)Y - (Yf)X.
\end{aligned}$$

Since X and Y are independent, it follows that $Xf = Yf = 0$. Q.E.D.

Proposition 14. *Let Σ be a level set of u. Suppose that two or more eigenvalues of S coincide on some open set $U \subset \Sigma$ (and hence on all leaves that pass through U). Then*

a) *the distribution $\mathscr{F}$ consisting of the span of the corresponding eigenvectors and the vector $\frac{\partial}{\partial u}$ is integrable, and*

b) *the common eigenvalue is constant on $\ell \cap U$, where ℓ is a leaf of the corresponding foliation.*

Proof. Suppose $\lambda_{k-1} < \lambda \equiv \lambda_k = \ldots = \lambda_{k+l} < \lambda_{k+l+1}$ on U. Then λ is C^1 on U, since S is C^1. For a) it suffices to show that $\left[X, \frac{\partial}{\partial u}\right] \in \mathscr{F}$ and $[X, Y] \in \mathscr{F}$ where X and Y are sections of $\mathscr{F}$ that are always tangent to the level sets of u. Since

$$\tilde{\nabla}_{\partial/\partial u}(SX) = (\tilde{\nabla}_{\partial/\partial u} S) X + S \tilde{\nabla}_{\partial/\partial u} X = -S^2 X + S \tilde{\nabla}_{\partial/\partial u} X \text{ (by Lemma 7e)}$$

and

$$\tilde{\nabla}_{\partial/\partial u}(SX) = \tilde{\nabla}_{\partial/\partial u}(\lambda X) = -\lambda^2 X + \lambda \tilde{\nabla}_{\partial/\partial u} X \quad \text{(by Corollary 8b)},$$

it follows that $S\tilde{\nabla}_{\partial/\partial u} X = \lambda \tilde{\nabla}_{\partial/\partial u} X$. Thus

$$S\left[X, \frac{\partial}{\partial u}\right] = S(SX - \tilde{\nabla}_{\partial/\partial u} X) = \lambda(SX - \tilde{\nabla}_{\partial/\partial u} X) = \lambda \left[X, \frac{\partial}{\partial u}\right].$$

To show that $[X, Y] \in \mathscr{F}$, let $Z \in T\Sigma$ with $SZ = \lambda' Z$, $\lambda' \neq \lambda$. Then

$$\begin{aligned}
&\lambda' H(u)(Z, [X, Y]) \\
&= H(u)\left(Z, \tilde{\nabla}_{[X,Y]} \frac{\partial}{\partial u}\right) && \text{(by Lemma 7a)} \\
&= H(u)(Z, \tilde{\nabla}_X(SY) - \tilde{\nabla}_Y(SX)) \\
&= H(u)(Z, (X\lambda) Y - (Y\lambda) X + \lambda[X, Y]) \\
&= \lambda H(u)(Z, [X, Y]) && \text{(since } H(u)(X, Z) = H(u)(Y, Z) = 0).
\end{aligned}$$

Thus $H(u)(Z, [X, Y]) = 0$. Letting Z range through the eigenvectors with eigenvalues different from λ, it follows that $S[X, Y] = \lambda[X, Y]$.

The proof of b) is the same as that of Proposition 13. Q.E.D.

If X and Y are vector fields tangent to a level set Σ, define

$$\nabla_X Y = \tilde{\nabla}_X Y + H(u)(X, Y) \frac{\partial}{\partial u}.$$

This is the Gauss equation in the present context. Since

$$du(\tilde{\nabla}_X Y) = X(du(Y)) - (\tilde{\nabla}_X du)(Y) = -H(u)(X, Y),$$

it follows that $\nabla_X Y \in T\Sigma$. It is easy to show that ∇ is a torsion-free connection on Σ. Computing the tangential and normal components of $\tilde{R}(X, Y) Z$, for $X, Y, Z \in T\Sigma$, and setting them equal to zero yields

$$R(X, Y) Z = H(u)(Y, Z) SX - H(u)(X, Z) SY \tag{2}$$

and

$$(\nabla_X H(u))(Y, Z) = (\nabla_Y H(u))(X, Z).$$

Here $H(u)$ is considered as a tensor on Σ and

$$(\nabla_X H(u))(Y, Z) = X(H(u)(Y, Z)) - H(u)(\nabla_X Y, Z) - H(u)(Y, \nabla_X Z).$$

Similarly, $(\nabla_X S)\, Y = \nabla_X(SY) - S(\nabla_X Y)$. Since

$$H(u)\,(\tilde{\nabla}_X Y, Z) = H(u)\,(\nabla_X Y, Z), \tag{3}$$

it follows that

$$\nabla_X H(u) = \tilde{\nabla}_X H(u)|_{T\Sigma}. \tag{4}$$

Since the importance of a particular level set Σ is minimal, the operator ∇ will usually be thought of as having for its domain and range the tensor fields on Ω that are annihilated when contracted with du or $\frac{\partial}{\partial u}$.

As the next two results show, the quantities ∇S and $\nabla H(u)$ are measures of the deviation of the solution u from the standard solution

$$u_s = r = \sqrt{\Sigma x_k^2}.$$

Theorem 15. *$\nabla S = 0$ on Ω if and only if the leaves of the foliation are parallel or meet in a single point when extended.*

Proof. First, suppose that the leaves are parallel or that they meet in a single point. By Proposition 12 and the observation following Definition 4, it follows that $S = kI\colon T\Sigma \to T\Sigma$, where k is a constant and Σ is a connected component of a level set. Then

$$(\nabla_X S)\,(Y) = \nabla_X(SY) - S(\nabla_X Y) = \nabla_X(k\,Y) - k\,\nabla_X Y = 0.$$

Conversely, suppose $\nabla S = 0$ on Ω. Let Σ be a connected component of a level set.

Claim. The eigenvalues of S are all equal and constant on Σ.

Proof of Claim. To obtain a contradiction, suppose that $\lambda \neq \lambda'$ at some point $p \in \Sigma$. Since S is C^1, its eigenvalues are C^1 in a neighborhood of almost every point of Σ. It can be assumed that λ and λ' are C^1 near p. Let $X, Y, Z \in T_p\Sigma$, all non-zero, with $SX = \lambda X$ and $SZ = \lambda' Z$. Extend X to a C^1 field near p so that it continues to be an eigenvector. It follows then, as in the proof of Proposition 14, that $\lambda' H(u)(Z, \nabla_Y X) = \lambda H(u)\,(Z, \nabla_Y X)$, and so

$$S(\nabla_Y X) = \lambda\, \nabla_Y X. \tag{5}$$

It follows from (2) that $R(X, Z)\, X = -H(u)(X, X)\, SZ$. The left side of this has eigenvalue λ by (5), whereas the right side has eigenvalue λ'. Thus both sides are zero and $\lambda' = 0$, since $H(u)(X, X) \neq 0$. Interchanging X and Z yields $\lambda = 0$, a contradiction.

It follows that (5) holds for all vectors tangent to Σ in a neighborhood of p. Then $S(\nabla_Y X) = \nabla_Y(SX) = (Y\lambda)\, X + \lambda\, \nabla_Y X$ implies that λ is constant near p. Since the eigenvalues are continuous, they are equal and constant on all of Σ, completing the proof of the claim.

We now have that for each connected component Σ of a level set, the map S has the form $S = k(\Sigma)\, I\colon T\Sigma \to T\Sigma$. The condition $S = 0$ is a closed condition on Ω. By the proof of the claim and Corollary 8b, $\nabla S \equiv 0$ implies that it is also an open condition. By connectedness, either $S \equiv 0$ on Ω, and the leaves are parallel, or $S \neq 0$ on Ω. In the latter case, consider the map $F: \Omega \to \mathbb{R}^{n+1}$, $p \to \Phi_{-1/k(\Sigma)}(p)$ for $p \in \Sigma$. By Proposition 12, F is constant on each Σ. But

$$F_*\left(\frac{\partial}{\partial u}\right) = \tilde{\nabla}_{\partial/\partial u}\left(p - \frac{1}{k(\Sigma)}\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u} - \left(\frac{\partial}{\partial u}\frac{1}{k(\Sigma)}\right)\frac{\partial}{\partial u} = 0$$

by Corollary 8b. Thus F is constant, and it follows that the leaves all meet at the image of F. Q.E.D.

Corollary 16. *Suppose that $\nabla S = 0$ on Ω, $S \neq 0$, and u is analytic. Then u is the restriction to Ω of a solution of the type constructed in Example 5a.*

There are also geometric conditions on Ω (e.g., convexity) that will guarantee that solutions satisfying $\nabla S \equiv 0$ and $S \neq 0$ will be of the type in Example 5a. As an example of a solution not of this type, consider the following:

Let $\Sigma \subset \mathbb{R}^2$ be a C^4 strictly convex loop that contains the semicircle $\{x^2 + y^2 = 4,\ x \geqq 0\}$ as an arc, and that does not meet the closed unit disk $\bar{\Delta}$. On $\mathbb{R}^2 - \{0\}$, let u_1 be the standard solution, and let u_2 be the solution given by the construction of Example 5a for which $u_2(0) = 0$ and $u_2(\Sigma) = 2$. Then $u_1 = u_2$ on $\{x \geqq 0\}$. Let $\Omega = \mathbb{R}^2 - (\{0\} \cup \{x^2 + y^2 = 1,\ x \leqq 0\})$, and let

$$u = \begin{cases} u_1 & \text{on } \Delta \cup \{x > 0\} \\ u_2 & \text{on } \{x < 0\} - \bar{\Delta}. \end{cases}$$

This solution has the property that the leaves, when extended, pass through $p = 0$, and so $\nabla S \equiv 0$, but it is not the restriction to Ω of any solution of the type in Example 5a, unless Σ is the circle $x^2 + y^2 = 4$.

Theorem 17. (First characterization of the standard solution.) *Suppose that some level set has a compact component, and that $\nabla H(u) = 0$ on Ω. Then, in appropriate affine coordinates and after replacing u by $au + b$, u is the restriction to Ω of the standard solution.*

Proof. Let $X, Y, Z \in T\Sigma$ be fields on an open subset of some level set Σ. Extend X, Y, Z to an open subset of Ω by parallel translation along the leaves of the foliation. Using (3), (4), Lemma 7, and $\nabla H(u) = 0$, it is easy to compute that

$$\begin{aligned} \frac{\partial}{\partial u} Z(H(u)(X, Y)) = &-H(u)(SX, \nabla_Z Y) - H(u)(S\nabla_Z X, Y) - H(u)(\nabla_{SZ} X, Y) \\ &- H(u)(X, \nabla_{SZ} Y), \end{aligned}$$

$$Z\frac{\partial}{\partial u}(H(u)(X, Y)) = -H(u)((\nabla_Z S)X, Y) - H(u)(S\nabla_Z X, Y) - H(u)(SX, \nabla_Z Y),$$

and

$$\begin{aligned} \left[\frac{\partial}{\partial u}, Z\right](H(u)(X, Y)) &= -(SZ)(H(u)(X, Y)) \\ &= -H(u)(\nabla_{SZ} X, Y) - H(u)(X, \nabla_{SZ} Y). \end{aligned}$$

It follows that

$$0=\left(\frac{\partial}{\partial u}Z-Z\frac{\partial}{\partial u}-\left[\frac{\partial}{\partial u},Z\right]\right)(H(u)(X,Y))=-H(u)((\nabla_Z S)X,Y).$$

Hence $\nabla S=0$ on Ω and the lines of the foliation are parallel or they meet in a single point p.

Let Σ' be the compact component of the level set in the hypothesis. A standard fact of differential geometry is that a connected, compact, strictly convex hypersurface in $\mathbb{R}^{n+1}$ must be a topological n-sphere. The leaves of the foliation cannot be parallel, for then some leaf would be tangent to Σ'. Similarly, p cannot be outside Σ' (that is, in the unbounded component of $\mathbb{R}^{n+1}-\Sigma'$). Thus, p is inside Σ', and the eigenvalue of S on Σ' is positive. Hence, the eigenvalue function $\lambda\colon \Omega\to\mathbb{R}$ is always positive. (As in Theorem 15, if $\lambda=0$ somewhere, then $\nabla S\equiv 0$ implies $\lambda\equiv 0$.) Without loss of generality, $u(\Sigma')=\lambda(\Sigma')=1$ (if not, replace u by $\lambda(\Sigma')(u-u(\Sigma'))+1$).

Define a Riemannian metric on Ω by $g=du^2+\frac{1}{\lambda}H(u)$.

Claim. $\tilde{\nabla}$ is compatable with g, that is, $\tilde{\nabla}g=0$.

Proof of Claim. Let X, Y, Z be fields that are always tangent to the level sets of u. Let W be an arbitrary field. Then

$$\tilde{\nabla}_X g=(\tilde{\nabla}_X du)\otimes du+du\otimes(\tilde{\nabla}_X du)+\frac{1}{\lambda}\tilde{\nabla}_X H(u).$$

Contracting with Y and Z,

$$(\tilde{\nabla}_X g)(Y,Z)=\frac{1}{\lambda}(\tilde{\nabla}_X H(u))(Y,Z)=\frac{1}{\lambda}(\nabla_X H(u))(Y,Z)=0,$$

by hypothesis and (4). Also

$$(\tilde{\nabla}_X g)\left(\frac{\partial}{\partial u},W\right)=H(u)(X,W)+\frac{1}{\lambda}(\tilde{\nabla}_X H(u))\left(\frac{\partial}{\partial u},W\right).$$

This last line is zero since

$$(\tilde{\nabla}_X H(u))\left(\frac{\partial}{\partial u},W\right)=(\tilde{\nabla}_{\partial/\partial u}H(u))(X,W)=-H(u)(SX,W)=-\lambda H(u)(X,W)$$

by Lemma 7f, h. Finally,

$$\tilde{\nabla}_{\partial/\partial u}g=H(u)+\frac{1}{\lambda}\tilde{\nabla}_{\partial/\partial u}H(u)=H(u)+\frac{1}{\lambda}(-\lambda H(u))=0,$$

by Lemma 7f, Corollary 8b, and the fact that du is parallel along the leaves. This completes the proof of the claim.

Since $\tilde{\nabla}$ is torsion-free, it is the unique Levi-Civita connection for the metric g. It follows that g can be defined on all of $\mathbb{R}^{n+1}$, since orthonormal frames can be translated unambiguously via $\tilde{\nabla}$. Then $(\mathbb{R}^{n+1}, g)$ is Euclidean space, $\frac{\partial}{\partial u}$ is the unit normal on each level set Σ, and $S\colon T\Sigma \to T\Sigma$ is the Weingarten map. Each point of Σ is umbilic ($S=\lambda(\Sigma)I$ on Σ), so Σ is a subset of the Euclidean sphere with center p and radius $\frac{1}{\lambda(\Sigma)}$ (see [12], p. 36). Choose affine coordinates on $\mathbb{R}^{n+1}$ so that p is the origin, and the g-sphere centered at p with radius R has equation $\sum x_k^2 = R^2$. Then $\frac{\partial}{\partial u} = \frac{\partial}{\partial r}$ and $du = dr$ on Ω, so $u = r + C$ on Ω. But Σ' is the sphere of radius 1, so $C=0$ and $u=r$ on Ω. (Note this implies that Σ' is the only component of $\{u=1\}$.) Q.E.D.

In the next two theorems the conditions of (1) are not assumed.

Theorem 18. (Second characterization of the standard solution.) *Let $\Omega \subset \mathbb{R}^{n+1}$ be open containing the origin and star-shaped with respect to the origin. Let u: $\Omega \to \mathbb{R}$ be continuous with $u(0)=0$, $u(x)>0$ for $x \neq 0$, and a C^2 solution of $\det H(u)=0$ on $\Omega - \{0\}$. Suppose that $\tau = f(u)$ is C^4 and strictly convex on a neighborhood V of 0, where $f\colon \mathbb{R} \to \mathbb{R}$ is C^4 and $f(0)=0$. Then, in appropriate affine coordinates, u is the standard solution of* (1) *and* $\tau = f(u) = a u^2 + O(u^4)$, $a>0$, *near* $u=0$.

Proof. The hypotheses clearly imply that τ has a local minimum at 0, and that for sufficiently small $\varepsilon>0$, the level sets $\{\tau=\varepsilon\}$ are strictly convex n-spheres containing the origin. These are also the level sets of u, so $\operatorname{rk} H(u) \geqq n$. Then $\det H(u)=0$ implies that u satisfies (1) on $V-\{0\}$. Since the leaves cross the level sets transversally, all leaves passing through points near 0 must meet at 0. Then, on a perhaps smaller punctured neighborhood of 0, u is a solution of the type in Example 5a, and is C^4 there, since its level sets are C^4. Also $S = \frac{1}{u} I$ on Σ_u for sufficiently small u.

Given $\det H(u)=0$, having u satisfy (1) is an open condition. On any ray starting at the origin it is also a closed condition, since $H(u)$ can be put in the form $\operatorname{diag}\left(0, \frac{1}{u}, \ldots, \frac{1}{u}\right)$ on any ray where (1) holds (this is a modification of Corollary 8c). If $\gamma(t)$ is a geodesic with $\gamma(0)=0$ and if (1) holds on γ for $0<t<r$, then $H(u)$ cannot drop rank at $\gamma(r)$, so (1a) holds. The foliation thus extends to $\gamma(r)$. Since $du \neq 0$ on the foliation, (1b) holds. The Hessian $H(u)$ cannot change signature, and so (1c) holds. In addition, the level set through $\gamma(r)$ is C^4, since it is determined by the level sets through $\gamma(t)$ for $t<r$ and the map Φ, which are all C^4. The solution continues to be of the type in Example 5a, and so it is C^4. Since Ω is star-shaped, each ray through 0 meets Ω in a connected segment or ray, and so (1) holds on all of Ω. In fact, the solution can be extended to all of $\mathbb{R}^{n+1}$.

Since du cannot be defined at 0 (in particular $du \neq 0$ at 0), and $d\tau = f'(u)\,du$ is continuous at 0, it follows that $f'(0)=0$. On $V-\{0\}$ we compute that

$$H(\tau) = f''(u)\,du^2 + f'(u)\,H(u) \tag{6}$$

and

(7) $$\tilde{\nabla}_{\partial/\partial u} H(\tau) = f'''(u)\, du^2 + \left(f''(u) - \frac{1}{u} f'(u)\right) H(u).$$

At 0, $H(\tau)$ is positive definite, and so $f''(0) > 0$ by (6).

Let ℓ and ℓ' be opposite leaves of the foliation in the sense that their union is an affine line through 0. Let $\frac{\partial}{\partial u_\ell}, \frac{\partial}{\partial u_{\ell'}} \in T_0 \mathbb{R}^{n+1}$ be the limiting values of $\frac{\partial}{\partial u}$ along ℓ and ℓ' at 0: $\frac{\partial}{\partial u_\ell} = -c \frac{\partial}{\partial u_{\ell'}}$ for some $c > 0$. By (6),

$$H(\tau)\left(\frac{\partial}{\partial u_\ell}, \frac{\partial}{\partial u_\ell}\right) = H(\tau)\left(\frac{\partial}{\partial u_{\ell'}}, \frac{\partial}{\partial u_{\ell'}}\right) = f''(0) \neq 0,$$

and hence $c = 1$. By (7), $(\tilde{\nabla}_{\partial/\partial u} H(\tau))\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = f'''(u)$. Evaluating this on ℓ and ℓ' and letting u tend to zero yields $f'''(0) = 0$. Hence $\tau = f(u) = \frac{1}{2} f''(0)\, u^2 + O(u^4)$.

Let p be the point on ℓ where $u(p) = 1$, and let $X_p \in T_p \Sigma_1$. Let $X = u X_p$ along ℓ. Then $\tilde{\nabla}_{\partial/\partial u} X = X_p = \frac{1}{u} X = SX = \tilde{\nabla}_X \frac{\partial}{\partial u}$, so $\left[X, \frac{\partial}{\partial u}\right] = 0$. Let Y and Z be global parallel fields that are tangent to the level sets along ℓ. Then

$$H(\tau)(Y, Z) = f''(u)\, du(Y)\, du(Z) + f'(u)\, H(u)(Y, Z).$$

Differentiating this one finds, along ℓ, that

(8) $$(\tilde{\nabla}_X H(\tau))(Y, Z) = f'(u)\, (\tilde{\nabla}_X H(u))(Y, Z).$$

Note that this tends to zero at the origin, since $X = 0$ there and τ is C^4.

Let $g(u) = (\tilde{\nabla}_X H(u))(Y, Z)$ along ℓ. Then

$$\begin{aligned} g'(u) &= (\tilde{\nabla}_{\partial/\partial u} \tilde{\nabla}_X H(u))(Y, Z) = (\tilde{\nabla}_X \tilde{\nabla}_{\partial/\partial u} H(u))(Y, Z) && \left(\text{since } \left[X, \frac{\partial}{\partial u}\right] = 0\right) \\ &= X\big((\tilde{\nabla}_{\partial/\partial g} H(u))(Y, Z)\big) = -X\left(\frac{1}{u} H(u)(Y, Z)\right) && \text{(by Lemma 7f)} \\ &= -\frac{1}{u} (\tilde{\nabla}_X H(u))(Y, Z) = -\frac{1}{u} g(u). \end{aligned}$$

Then $g(u) = \frac{c}{u}$ for some $c \in \mathbb{R}$, and $(\tilde{\nabla}_X H(\tau))(Y, Z) = \frac{c}{u} f'(u) = c f''(0) + O(u^2)$. It follows from the remark after (8) that $c = 0$. Hence $(\nabla_X H(u))(Y, Z) = (\tilde{\nabla}_X H(u))(Y, Z) = 0$, that is, $\nabla H(u) = 0$ on $\mathbb{R}^{n+1} - \{0\}$.

The result now follows from Theorem 17. Note that perturbing $f(u)$ by $O(u^4)$ does not affect any of the proof, so there are no additional restrictions on $f(u)$. Q.E.D.

If u is any solution of the type in Example 5a in a neighborhood of 0, and $f\colon \mathbb{R} \to \mathbb{R}$ is any function with $f'(x) > 0$ and $f''(x) > 0$ for $x > 0$, then $f(u)$ is strictly convex away from 0. The point of the theorem is to see the implication of extending the strict convexity to the origin.

As an application, we give a proof of the following theorem due to Burns [3]. It is the real version of Stoll's characterization of balls in $\mathbb{C}^n$. (See Theorem H of the introduction.)

Theorem 19. *Let M be a connected $(n+1)$-dimensional affine manifold. Let τ: $M \to [0, R^2)$, $0 < R \leqq \infty$ be C^4, proper, and strictly convex. Suppose that $u = \sqrt{\tau}$ satisfies* $\det H(u) = 0$ *on* $\{\tau \neq 0\}$. *Then there is an affine diffeomorphism F: $B_R(0) \to M$, where $B_R(0) = \{x \in \mathbb{R}^{n+1}: |x| < R\}$, such that $u \circ F$ is the standard solution of* (1).

Proof. From $d\tau = 2u\,du$ and $H(\tau) = 2du^2 + 2u\,H(u)$, it follows that u satisfies (1) on $\{u \neq 0\}$. Since $\{0 \leqq \tau \leqq r^2\}$ is compact in M for $r < R$, τ must take a minimum on this set. It clearly doesn't take it on the boundary $\{\tau = r^2\}$, so it must take it at an interior point. Since $d\tau = 2u\ du \neq 0$ for $u \neq 0$, the minimum must occur at a point where $\tau = u = 0$. Let $p \in \{\tau = 0\}$. Since τ is strictly convex, p is an isolated critical point. On a small star-shaped neighborhood V of p, Theorem 18 applies, and there is an affine coordinate chart G: $V \to \mathbb{R}^{n+1}$ such that $G(p) = 0$ and u is the standard solution in these coordinates.

Let ℓ be a leaf of the foliation that passes through p when extended. For $t > 0$ let $\gamma(t)$ be the point on ℓ where $u(\gamma(t)) = t$; γ is defined for all small values of t. Then γ is a geodesic, and $\dot{\gamma}(t) = \left.\frac{\partial}{\partial u}\right|_{\gamma(t)}$. If γ is defined for all $t < r < R$, then γ is defined for $t = r$ by the compactness of $\{0 \leqq u \leqq r\}$. Since geodesics can always be extended, $\gamma(t)$ is defined for some values of $t > r$. It follows that $\gamma(t)$ is defined for all $0 < t < R$, and that u takes on all the values in $(0, R)$ on each leaf that passes through p. The union of the leaves that meet at p is therefore a maximal open connected set in M, and so it is all of M by connectedness. (It follows that p is the only point in $\{\tau = 0\}$.)

Let Exp_p: $T_p M \to M$ be the exponential map for the affine structure. It is affine, and $\mathrm{Exp}_p^{-1} \circ G^{-1}$ induces an affine coordinate system on $T_p M$. Let $B_R(0)$ be the ball of radius R in these coordinates. The preceding paragraph proves that the domain of Exp_p is precisely $B_R(0)$, and, in fact, that Exp_p: $B_R(0) \to M$ is a diffeomorphism (since it is one-to-one). Then $F = \mathrm{Exp}_p$ is the desired map. Q.E.D.

Applications to the Cauchy Problem for $\det\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = 0$

Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface, and let φ, ψ: $M \to \mathbb{R}$. We seek a function u defined on a neighborhood of M that satisfies

$$u = \varphi \text{ and } Nu = \psi \text{ on } M, \quad \text{and} \quad \det H(u) = 0 \text{ near } M, \tag{9}$$

where N is the unit normal to M.

In general, solutions of (9) may not exist even for C^∞ or C^ω data. However conditions can be placed on M, φ, and ψ that will guarantee existence and uniqueness of solutions to (9) by forcing properties similar to (1a, b).

It should be noted that Definitions 2, 3, and 4, Examples 5, and Proposition 6 are still valid when (1) is replaced by (1a, b). Note that conditions (1a, b) imply that $H(u)$ is definite on $T\Sigma$ for any level set Σ of u.

In general, solutions of $\operatorname{rk} H(u)=n$ do not need to satisfy (1b), since it is possible for a level set to contain a leaf of the foliation.

Let r be the distance function for M: $r(p+sN_p)=s$ for $p\in M$ and $s\in\mathbb{R}$ near zero. The Gauss equation defines a connection ∇ on TM:

$$\nabla_X Y=\tilde{\nabla}_X Y+H(r)(X, Y)N.$$

The Weingarten map for M is given by

$$W\colon TM\to TM, \qquad W(X)=\tilde{\nabla}_X N. \qquad \text{(See [12].)}$$

The Hessian of a C^2 function $\varphi\colon M\to\mathbb{R}$ is defined by

$$H_M(\varphi)(X, Y)=(\nabla_X d\varphi)(Y)=X(d\varphi(Y))-d\varphi(\nabla_X Y).$$

Here $X, Y\in TM$ and Y is extended in an arbitrary manner to be a local C^1 field tangent to M.

The next result should be compared with Proposition 1.5 of [1].

Proposition 20. *Let M and φ be C^3, and ψ be C^2. If $H_M(\varphi)+\psi H(r)$ is definite on TM, then there exists a unique C^2 solution u of* (9).

Proof. Suppose that such a function u exists. If $X\in TM$, a few simple computations show that $du(X)=d\varphi(X)$, $du(N)=\psi$, $H(u)(X,N)=d\psi(X)-d\varphi(WX)$, and $H(u)=H_M(\varphi)+\psi H(r)$ on TM. It follows that $\operatorname{rk} H(u)=n$, and that the leaves of the foliation are transverse to M (in [1], M is said to be *non-characteristic for u*). At each point of M there is a unique vector $Z\in TM$ so that $Z+N$ is tangent to the foliation. Then for $X\in TM$ it follows that $H(u)(X,Z+N)=0$, that is,

$$(H_M(\varphi)+\psi H(r))(X,Z)=d\varphi(WX)-d\psi(X).$$

This condition, which is free of the unknown function u, uniquely determines Z as a C^1 vector field tangent to M. The function u can then be defined by

$$u(p+r(N+Z))=\varphi(p)+r(d\varphi(Z)+\psi). \tag{10}$$

(Compare this with the construction in Example 5b.) This is C^2 by an argument similar to that of Proposition 6b. It is an easy matter to show that u satisfies (9). Note that $du(N+Z)=d\varphi(Z)+\psi$. When this is non-zero, the solution u satisfies (1a, b), and $\dfrac{\partial}{\partial u}=\dfrac{Z+N}{d\varphi(Z)+\psi}$. When $d\varphi(Z)+\psi=0$, the leaf of the foliation is contained in a level set. Q.E.D.

The solution u constructed here depends smoothly on the Cauchy data φ and ψ in the following sense. Suppose that M is C^k and that φ_t and ψ_t, $|t|<\varepsilon$, are one-parameter families of Cauchy data of regularity C^k and C^{k-1} respectively, $k \geqq 3$. If φ_0 and ψ_0 satisfy the hypothesis of Proposition 20, then so will φ_t and ψ_t for all $t \in (-\varepsilon, \varepsilon)$, when ε is chosen sufficiently small. For each fixed t there is a solution $u_t \in C^{k-1}$ of (9) given by (10). From the construction in the proof, it is seen that the family of solutions u_t is at least C^{k-2}. Making the dependence of u, φ, ψ, and Z on t explicit in (10), differentiating with respect to t, and using

$$du_t\left(\frac{dZ_t}{dt}\right) = d\varphi_t\left(\frac{dZ_t}{dt}\right) \qquad \left(\text{since } \frac{dZ_t}{dt} \in TM\right),$$

it follows that

$$\left(\frac{d}{dt} u_t\right)(p + r(N + Z_t)) = \frac{d}{dt}\varphi_t(p) + r\left(\left(\frac{d}{dt}\varphi_t\right)(Z_t) + \frac{d\psi_t}{dt}\right).$$

Thus $\frac{du_t}{dt}$ is C^{k-2}, and so u_t is C^{k-1} as a one-parameter family.

The following special case is useful in guaranteeing solutions of (1) with nice properties.

Corollary 21. *Let Σ be a C^2 hypersurface in $\mathbb{R}^{n+1}$ with non-vanishing Gaussian curvature. Let ω be a C^2 one-form along Σ with $\omega = 0$ on $T\Sigma$ and $\omega(X) \neq 0$ for vectors X transverse to Σ. Then there exists a unique C^2 solution u of* (1 a, b) *on a neighborhood of Σ with $u = 0$ and $du = \omega$ on Σ.*

Proof. Let $M = \Sigma$, $\varphi = 0$, and $\psi = \omega(N) \neq 0$ in Proposition 20. The curvature condition implies that $H(r)$ is definite. The surface only needs to be C^2 since the operators H_Σ and W do not enter. Q.E.D.

In the corollary, the tangent space to the leaf through $p \in \Sigma$ is given by $\ell = \bigcap_{X \in T_p\Sigma} \tilde{\nabla}_X \omega$, which is one-dimensional and transverse to Σ by the curvature assumption. One gets that $H(u) = \tilde{\nabla}\omega$ along Σ, and that $\frac{\partial}{\partial u}$ is the unique vector in ℓ with $\omega\left(\frac{\partial}{\partial u}\right) = 1$. If Σ is strictly convex, then the eigenvalues of $S\colon T\Sigma \to T\Sigma$ are defined, and so the maximum domain for u can be written in terms of the Cauchy data. In particular, explicit conditions on the Cauchy data exist that will guarantee that the solution u be exterior or strictly exterior.

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Received March 12, 1986