Math 225 **Final Exam** **Name:**

Wabash College, Arctic Campus

13 December 2010 200 Points

"Show enough work to justify your answers."

READ CAREFULLY: Do at least **ten** problems. If you work on more than ten, you will get credit for the best ten, however *you must do at least one max/min problem, at least one multiple integral problem, and at least one path integral problem* (these sections are marked). Note that some problems have multiple parts. (20 points each)

READ CAREFULLY: You may use *Mathematica* to help you think on any problem. You may use it as part of your solution **only** on the indicated problems. If you need help with *Mathematica*, please ask.

Part I, Max/Min. Do at least one problem from this part.

1. Let $f(x, y, z) = y^2 - 2xy + \frac{x^3}{3} + z^2$.
   (a) Find all critical points of f (there is more than one). (8 points)

   (b) For each critical point, determine if it is a local max, local min, or saddle point. (12 points)

2. Find the maximum and minimum values of $f(x, y) = x^2 - 2x + y^2$ and their locations on the disk $x^2 + y^2 \leq 4$.

3. Let C be the curve of intersection of $z^2 = x^2 + 4y^2$ and $x + 2y + 3z = 6$. The point of this problem is to find the points on C that are closest to and farthest from the origin. You may use *Mathematica* for all calculations. See the templates in MaxMinFest.nb in the N:/Math/Math225 folder.
   (a) Clearly write the algebraic, scalar equations that you would need to solve if you were to find the critical points by hand. (7 points)
   (b) Use *Mathematica* to solve the equations. *Mathematica* can get the exact values, but they are horrible, so get numercal approximations. Write the solutions here. Ignore those involving complex numbers. (7 points)
   (c) Make your conclusions. Since I won't see your *Mathematica* work, give enough information to justify your conclusions. (6 points)

Part II, Multiple integrals. Do at least one problem from this part.

4. Let S be the solid region given by $0 \le z \le y \le x \le 1$. Find the volume of S and the x-coordinate of its centroid.

5. Let S be the solid region that is inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cone $z^2 = x^2 + y^2$, $z \ge 0$.

   (a) Express the following integral in ready-to-evaluate form in either cylindrical or spherical coordinates.
   $$\iiint_S \frac{1}{x^2 + y^2 + z^2}\, dV$$

   (b) Evaluate your integral in part a). If it is in cylindrical coordinates, you may use *Mathematica*. If it is in spherical coordinates, evaluate it by hand.

6. Let D be the parallelogram enclosed by the lines $2x + y = 1$, $2x + y = 4$, $x - y = -1$, and $x - y = 1$. Evaluate the following integral by changing it to (u, v) coordinates where $u = 2x + y$ and $v = x - y$. Recall that $\frac{\partial(x,y)}{\partial(u,v)} = 1/\frac{\partial(u,v)}{\partial(x,y)}$.
   $$\iint_D (2x + y)^2 e^{x-y}\, dA$$

Part III, Path integrals. Do at least one problem from this part.

7. Let C be the circle $x^2 + y^2 = 4$ oriented counterclockwise. Evaluate the following.
   $$\int_C -(x^2 + y^2)y\, dx + (x^2 + y^2)x\, dy$$

8. Let C be the boundary of the quarter disk $x^2 + y^2 = 4$, $x, y \ge 0$ (this has three parts). Evaluate the following.
   $$\int_C x\, ds$$

9. Let C be the parallogram with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, $(1, 2)$, oriented counterclockwise. Use Green's Theorem to evaluate the following.
   $$\int_C (3y + \sin x^2)\, dx + (5x - e^{y^2})\, dy$$

Part IV, Other problems.

10. Consider the plane determined by the points $A(5,0,0)$, $B(6,0,2)$, and $C(5,1,3)$.

    (a) Find an implicit equation of the plane.

    (b) Let ℓ be the line passing through the points $P(1,1,0)$ and $Q(2,0,-2)$. Find the point where ℓ intersects the plane. Suggestion: Parameterize the line.

11. Suppose $\mathbf{v} : \mathbb{R} \to \mathbb{R}^n$ is a differentiable vector function.

    (a) Compute an expression for $\frac{d}{dt} \|\mathbf{v}(t)\|^2$ in terms of $\mathbf{v}(t)$ and $\mathbf{v}'(t)$. For full credit, your computation must be coordinate-free.

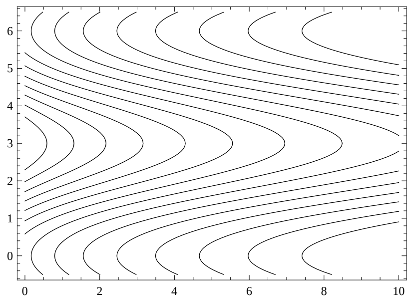
    (b) Use your answer in part a) to prove that $\mathbf{v}(t)$ has constant length if and only if $\mathbf{v}(t)$ and $\mathbf{v}'(t)$ are always perpendicular.

12. Level curves of a function f are shown.

    (a) Draw the four flow curves of ∇f that pass through $(0,2)$, $(0,3)$, $(0,4)$, and $(0,5)$. (16 points)

    (b) Draw the flow curve of $\nabla f_\perp$ that passes through $(0,1)$. (4 points)

    There will be very little partial credit on this problem.



13. Let S be the surface in $\mathbb{R}^3$ given by $yz^3 + x^2z = 1$. Observe that the point $P(2,-3,1)$ is on S.

    (a) Find an implicit equation of the plane tangent to S at P. (10 points)

    (b) Use the Implicit Function Theorem to explain why S determines z as some differentiable function of x and y, $z = g(x,y)$, near P. (5 points)

    (c) Find the best affine (HS linear) approximation of g at $(x,y) = (2,-3)$. (This is closely related to the equation of the plane tangent to S at P.) (5 points)

14. Let $f(x, y) = x^2 y$, $P(1, 2)$, and $\mathbf{v} = -\mathbf{i} + \mathbf{j}$. Compute $D_{\mathbf{v}} f(P)$ in two ways. (10 points each)

   (a) Using the definition of $D_{\mathbf{v}} f(P)$.

   (b) Using the gradient of f.

15. Let $F(x, y, z) = (y^2, xz, \ln y)$ and $P(2, 1, 3)$. Let L be the best affine (HS linear) approximation to F near P.

   (a) Give a formula in coordinates for $L(x, y, z)$. (15 points)

   (b) What is $L(2.2, .9, 3.1)$? (5 points)

16. For each vector field, determine if it is a gradient. If it is, find a function f so that the vector field equals ∇f.

   (a) $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j}$

   (b) $\mathbf{G} = xy\mathbf{i} + x^2\mathbf{j}$

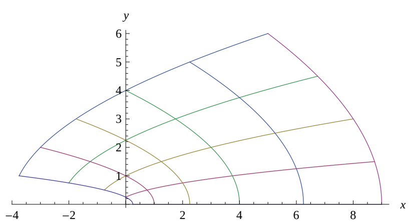
17. Consider coordinates (s, t) given by $(x, y) = F(s, t) = (s^2 - t^2, st)$. Some of the coordinate curves are shown below in the (x, y)-plane.

   (a) Plot the point $P = F(2, 1)$. (2 points)

   (b) Compute formulas for the unnormalized coordinate vector fields $\mathbf{v}_s$ and $\mathbf{v}_t$. (8 points)

   (c) Compute $\mathbf{v}_s$ and $\mathbf{v}_t$ at P and add them to the picture. Draw them to scale and label. (5 points)

   (d) Clearly indicate which coordinate curves are s-curves (on which t is constant) and which are t-curves (on which s is constant). (5 points)



Have a good break!

Partial answers and hints.

1. There are two critical points. One is a saddle; one is a local min.
2. The min is -1; the max is 8.
3. Closest: $(0.40716, 0.676633, 1.41319)$. Farthest: $(-3.12701, -0.992902, 3.70427)$.
4. $V = 1/6$, $\bar{x} = 3/4$
5. Try both coordinate systems. $2\pi(2 - \sqrt{2})$
6. $7(e - 1/e)$
7. 32π. There is a way to do this that involves very little computation.
8. 6
9. 16
10. (a) $2x + 3x - z = 10$ (b) $(6, -4, -10)$
13. (a) $4(x-2) + (y+3) - 5(z-1) = 0$
    (c) Solve (a) for z to get $\ell(x, y) = 1 + \frac{4}{5}(x-2) + \frac{1}{5}(y+3)$.
14. -3
15. (b) $(.8, 6.8, -.1)$
16. One of them is the gradient of x^2y. The other is not a gradient.