Math 225 **Final Exam** **Name:**

15 December 2009 200 Points

"Show enough work to justify your answers."

READ CAREFULLY: Do any **ten** problems. If you work on more than ten, you will get credit for the best ten. Note that some problems have multiple parts. (20 points each)

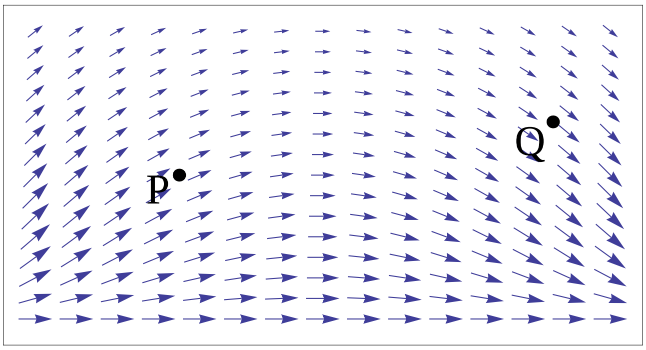
READ CAREFULLY: Do each problem on a separate sheet of paper, except as noted. Avoid writing anything in the upper left corner where the staple will go. Put problem numbers in the upper right corners.

READ CAREFULLY: You may use *Mathematica* to help you think on any problem. You may use it as part of your solution **only** on the indicated problems.

1. The gradient vectors of a function $f(x, y)$ are shown at a number of points in the xy-plane. Draw directly on the picture on the exam.

   (a) Clearly indicate the level curve of f that passes through the point P.

   (b) Suppose that an insect starts at the point Q and always moves in the direction f decreases most quickly. Clearly indicate the path of the insect and its direction of motion.



2. Given $A(3, 0, 2)$, $B(-2, 1, 2)$, $C(3, 1, 0)$, and $D(0, 13, 3)$.

   (a) Find a parametric or non-parametric equation of the plane passing through A, B, and C.

   (b) Find the point on the plane closest to D.

3. Consider the parabola $y = x^2$.

   (a) Determine the curvature of this curve as a function of x.

   (b) Determine the center and radius of the osculating circle of the curve at $x = 0$.

READ CAREFULLY: You may use *Mathematica* as part of your solution only on the next three problems. There is a template *Mathematica* file in the folder N:/Math/Math225/Foote for this exam. When you are finished, send your *Mathematica* file to me by e-mail. Check to be sure I received your e-mail before you leave.

4. Let C be the curve $\gamma(t) = (t^2, t^3)$ for $0 \le t \le 2$. Express $\int_C (x^2 + y^2)\, ds$ as a ready-to-evaluate integral. Evaluate it using *Mathematica*. Give the exact answer and a numerical approximation.

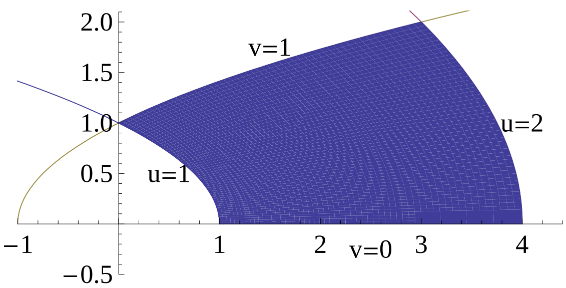
5. Let $f(x, y, z) = xyz$. The point of this problem is to find the maximum and minimum values of f and their locations on the curve of intersection of the ellipsoid $36x^2 + 9y^2 + 4z^2 = 36$ and the plane $6x + 3y + 2z = 6$.

   (a) Clearly write down the algebraic, scalar equations you would need to solve if you were to completely solve this problem by hand. If you are unsure what I mean by this, be sure to ask.

   (b) Use *Mathematica* to solve the equations in (a).

   (c) Clearly state your results in a complete sentence. Give exact answers.

6. Consider $\iint_R (x + y^2)\, dx\, dy$, where R is the region bounded by the curves $x = 1 - y^2$, $x = 4 - y^2/4$, $x = y^2 - 1$, and $y = 0$. Rewrite this integral in ready-to-evaluate form in the coordinates (u, v), where $x = u^2 - v^2$ and $y = uv$. You may use *Mathematica* to evaluate it (although it would be easy to do by hand). Give the exact answer.



7. The acceleration formula $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$ is valid for a C^2 curve in $\mathbf{R^3}$. Use it to prove the curvature formula

$$\kappa = \frac{||\mathbf{a} \times \mathbf{v}||}{||\mathbf{v}||^3}.$$

   (Note: If you are comfortable with the proof of the analogous curvature formula in $\mathbf{R^2}$, you can do this! Use a cross product instead of perp and dot. You need to know that $\kappa \ge 0$ in $\mathbf{R^3}$ and that $\mathbf{N}$ is a unit vector perpendicular to $\mathbf{T}$, even though it can't be defined using the perp operator. Remember there is no perp operator in $\mathbf{R^3}$.)

8. Find a point on the graph of $z = x^2 + 2y^2 + 1$ where the position vector is tangent to the graph. (There are infinitely many such points, forming a curve on the graph. You need to find just one.)

9. Let $f(x, y)$ be the distance from (x, y) to $(-2, 1)$.

   (a) Compute ∇f.

   (b) Show that if $(x, y) \neq (-2, 1)$, then $\nabla f(x, y)$ is the unit vector at (x, y) pointing directly away from $(-2, 1)$.

10. Let $g(x, y, z) = \min\{x, y, z\}$. Evaluate $\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g(x, y, z)\,dx\,dy\,dz$.

11. Suppose R is a bounded region in $\mathbf{R^2}$ with a "nice" boundary curve C (piecewise continuously differentiable).

   (a) Use Green's Theorem to explain why $\frac{1}{2}\oint_C -y\,dx + x\,dy$ is the area of R when C is oriented counterclockwise.

   (b) Now suppose R is the interior of the ellipse $4x^2 + 25y^2 = 100$. Use the path integral formula in part a) to find the area of the ellipse. Do all computations by hand.

12. Suppose that $f : \mathbf{R^2} \to \mathbf{R}$ is a C^2 function. Prove that $\text{curl}(\nabla f) = 0$.

13. Consider the two vector fields.

$$\mathbf{F} = (y^2 - 3x^2 + 2)\mathbf{i} + (2xy + \cos 3y)\mathbf{j} \qquad \mathbf{G} = (y^2 - x^2 + 3)\mathbf{i} + (xy + \cos 2y)\mathbf{j}$$

   (a) Show that one of the vector fields is conservative and the other is not.

   (b) Find a potential function for the conservative field.

14. If mass density is given by $\delta(x, y, z) = z^2$, find the mass of the solid that is inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

15. Let $f(x, y, z) = x^2 + \sin(xy) + y^2/2 + z^2$. Show that the origin is a critical point and determine if f has a local maximum, local minimum, or saddle point there.

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

Have a good break!

Selected answers and hints.

2. The closest point is $(-2, 3, -2)$. There are two approaches to finding the closest point. One is to use projection. The other is to parameterize the line through D that is perpendicular to the plane.

3. $\kappa = 2/(1 + 4x^2)^{3/2}$

4. The numerical approximation is 241.369.

5. The maximum value is 0. One point where it occurs is $(0, 2, 0)$. The minimum value is $-8/9$. One point where it occurs is $(2/3, 4/3, -1)$.

6. 256/15. The most common error is to forget the distortion factor.

8. One such point is $(-1, 0, 2)$. Note: the gradient of $x^2 + 2y^2 + 1$ is not perpendicular to the graph of $z = x^2 + 2y^2 + 1$.

10. -4

11. The area of the ellipse is 10π.

13. Check your answer to (b) by taking its gradient.

14. $12\pi\sqrt{3}/5$. Choose your coordinate system wisely! The equation of a sphere in cylindrical coordinates is nice. The equation of a cylinder in spherical coordinates is not so nice!

15. The origin is a local minimum.