**Math 225** **Final Exam** **Name:**

12 December 2007 200 Points

*"Show enough work to justify your answers."*

**READ CAREFULLY:** Do any **ten** problems. If you work on more than ten, you will get credit for the best ten. Note that some problems have multiple parts. (20 points each)

**READ CAREFULLY:** You may use *Mathematica* to help you think on any problem. You may use it as part of your solution only on the indicated problems. Save your *Mathematica* work. Indicate the problem numbers. Send your *Mathematica* file to me by e-mail.

**READ CAREFULLY:** Please do each problem on a separate sheet of paper, except as noted. Avoid writing anything in the upper left corner where the staple will go.

1. Given $P(3, -1, 2)$, $Q(0, 0, 2)$, $R(1, -1, 3)$, and $S(5, -3, 2)$,

   (a) Find a non-parametric equation of the plane passing through $P$, $Q$, and $R$, and

   (b) Find the distance from $S$ to the plane.

2. State and prove the formula that expresses the acceleration of a $C^2$ curve as a linear combination of its unit tangent and normal vectors.

3. If $f : \mathbf{R} \to \mathbf{R}$ is differentiable at $a \in \mathbf{R}$, the best linear approximation (in the high school sense) of $f$ near $a$ is, of course,

$$f(x) \approx \ell(x) = f(a) + f'(a)(x - a).$$

   In this case, the derivative is represented by a scalar, and the product of the derivative and the displacement $x - a$ is multiplication of numbers. Write the corresponding formulas for

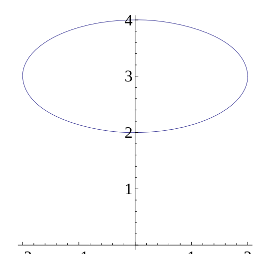
   (a) A curve $\gamma : \mathbf{R} \to \mathbf{E^n}$ that is differentiable at $a \in \mathbf{R}$,

   (b) A scalar function $f : E^n \to \mathbf{R}$ that is differentiable at $P \in E^n$, and

   (c) A mapping $F : \mathbf{R^n} \to \mathbf{R^k}$ that is differentiable at $P \in \mathbf{R^n}$.

   In each case, describe the mathematical object that represents the derivative and the type of product between the derivative and the displacement.

4. Find the points on the ellipse $x^2 + 4(y - 3)^2 = 4$ where the position vector from the origin is tangent to the ellipse. You may use *Mathematica* for the algebra, although it can be done by hand.



5. Let $T(\theta,\phi) = 3\mathbf{u}(\theta) + \mathbf{v}(\theta,\phi)$, where $\mathbf{u}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and $\mathbf{v}(\theta,\phi) = \cos\phi\,\mathbf{u}(\theta) + \sin\phi\,\mathbf{k}$. This is the torus map we have used several times. Find an equation, either parametric or non-parametric, of the plane tangent to the image of $T$, that is, tangent to the torus, at the point where $(\theta,\phi) = (\pi/4, \pi/4)$. (Think about what kind of geometric information you get about the tangent plane from the partial derivatives.)

6. Suppose that $w = g(u,v)$ is a differentiable function of $u = x/y$ and $v = z/y$. Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0.$$

7. (a) Define what it means for a curve $\gamma$ to be a flow curve of a vector field $\mathbf{F}$. (8 points)
   (b) Consider the vector field on $\mathbf{R}^2$ given by $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. Use "guess and check" methods to find formulas of two flow curves of $\mathbf{F}$, one passing through $(3,0)$, and one passing through $(0,5)$. Suggestion: Sketch enough of the vector field to get an idea of what it looks like. Don't forget the "check" part of "guess and check"! (12 points)

8. Consider polar coordinates in the $xy$-plane.
   (a) Draw a representative sample of coordinate curves. Label each curve with its polar coordinate equation (some variable equals a constant).
   (b) Compute formulas for the non-normalized coordinate vector fields $\mathbf{v}_r$ and $\mathbf{v}_\theta$.
   (c) Draw $\mathbf{v}_r$ and $\mathbf{v}_\theta$ on your coordinate curves at a minimum of four different locations. Your vectors should be to scale.

9. Logarithmic polar coordinates $(\rho,\theta)$ in $\mathbf{R}^2$ are given by $x = e^\rho\cos\theta$ and $y = e^\rho\sin\theta$. These are similar to regular polar coordinates except that $\rho = \ln r$. If $f$ is a function and $\mathbf{F} = P\mathbf{u}_\rho + Q\mathbf{u}_\theta$ is a vector field, both being functions of $\rho$ and $\theta$, then $\nabla f$ and $\operatorname{div}\mathbf{F}$ are given by

$$\nabla f = e^{-\rho}\frac{\partial f}{\partial \rho}\mathbf{u}_\rho + e^{-\rho}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta \qquad \text{and} \qquad \operatorname{div}\mathbf{F} = e^{-\rho}\left(\frac{\partial P}{\partial \rho} + P + \frac{\partial Q}{\partial \theta}\right),$$

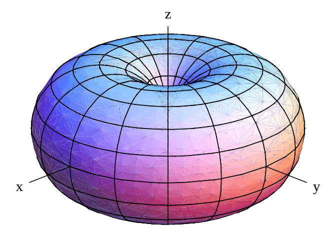
where $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$ are the unit coordinate vectors in this coordinate system. Compute $\Delta f = \nabla^2 f$ in this coordinate system.

10. Let $f(x,y,z) = kx^2 + ky^2 + 2yz - z^2$. This function has a critical point at $(0,0,0)$. You do not need to verify this. For what values of $k$ does
   (a) $f$ have a nondegenerate local minimum at $(0,0,0)$?
   (b) $f$ have a nondegenerate local maximum at $(0,0,0)$?
   (c) $f$ have a saddle point at $(0,0,0)$?
   (d) The second derivative test fail?

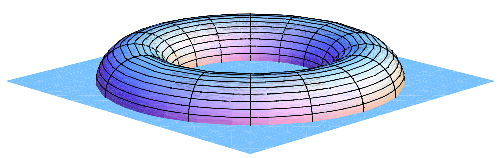
Suggestion: Start with part (d). Then consider the intervals on which (d) does not hold.

11. Find the maximum and minimum values of $f(x, y, z) = x^2 - 2x + y^2 + 4y + z^2 - 2z$ and their locations in the solid ball $x^2 + y^2 + z^2 \leq 9$. You may use *Mathematica* to do the algebra.

12. The surface $\rho = \sin\phi$ (spherical coordinates) is pictured. Note that any plane that contains the $z$-axis slices this surface in two circles. The circles are tangent to each other and to the $z$-axis at the origin. Find the volume of the solid contained by this surface. You may use *Mathematica*.




13. The $z$-coordinate of the centroid of a solid $S$ in $\mathbf{R^3}$ is given by $\bar{z} = \frac{1}{V}\iiint_S z\,dV$, where $V$ is the volume of $S$. Let $S$ be the upper half (above the $xy$-plane) of the solid torus given in cylindrical coordinates as $(r-3)^2 + z^2 = 1$ (this is the same torus as in the last problem set). Compute $\bar{z}$ for this solid. You may use *Mathematica*. (Note that $\bar{x} = \bar{y} = 0$ by symmetry.)



14. Evaluate $\int_C r^2\,ds$ where $C$ is the graph of $y = \cos x$ in $\mathbf{R^2}$ from $x = -\pi/2$ to $x = \pi/2$ and $r$ is distance to the origin. You may use *Mathematica* (you may need to use **NIntegrate**).

15. Consider the vector field on $\mathbf{R^2}$ given by $\mathbf{F} = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$. Suppose that $C$ is a simple, piecewise $C^1$, loop going around the origin counterclockwise. The point of this problem is to evaluate $\int_C \mathbf{F} \cdot dX$.

    (a) Suppose that $\tilde{C}$ is the circle of radius $r$ centered at the origin oriented counterclockwise. Evaluate $\int_{\tilde{C}} \mathbf{F} \cdot dX$.

    (b) Show that $\int_C \mathbf{F} \cdot dX = \int_{\tilde{C}} \mathbf{F} \cdot dX$ by applying Green's Theorem to the annular region between $C$ and $\tilde{C}$.

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

*Have a good break!*

Selected answers and hints.

1. (b) $2\sqrt{14}/7$

3. (a) $\gamma(t) \approx \gamma(a) + (t-a)\gamma'(a)$
   The derivative is a vector, and the product between $\gamma'(a)$ and $t-a$ is scalar multiplication.

4. $(\pm 4\sqrt{2}/3, 8/3)$ This can be done either by parameterizing the ellipse or by considering the ellipse to be a level curve of a function. Using an appropriate derivative, write a condition that says the position vector is tangent to the curve.

5. Parametric: $(x, y, z) = \left(\frac{3\sqrt{2}}{2} + \frac{1}{2} - t(1+3\sqrt{2}) - s, \frac{3\sqrt{2}}{2} + \frac{1}{2} + t(1+3\sqrt{2}) - s, \frac{\sqrt{2}}{2} + s\sqrt{2}\right)$

7. (b) $\gamma(t) = (5\cos t, 5\sin t)$ passes through $(0, 5)$

9. $\nabla^2 f = e^{-2\rho}\left(\frac{\partial^2 f}{\partial \rho^2} + \frac{\partial^2 f}{\partial \theta^2}\right)$

10. (a) This doesn't happen for any $k$.
    (c) This happens when $k > -1$ and $k \neq 0$.
    (d) The test fails when $k$ is 0 or $-1$.

11. Maximum: $9 + 6\sqrt{6}$. Minimum: $-6$. If you get a different minimum, you may have missed one of the critical points.

12. $\pi^2/4$

13. $4/(3\pi)$

14. Approx 5.21081

15. $2\pi$