Math 225 **Final Exam** **Name:**

13 December 2006 200 Points

"Show enough work to justify your answers."

READ CAREFULLY: Do any **ten** problems. If you work on more than ten, you will get credit for the best ten. Note that some problems have multiple parts. (20 points each)

You may use *Mathematica* to help you think on any problem. You may use it as part of your solution only on one problem, as indicated.

Please do each problem on a separate sheet of paper, except as noted. Avoid writing anything in the upper left corner where the staple will go.

1. Let L be the line passing through $P(-1, 1, 2)$ parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Find the point on L closest to $Q(0, 5, 2)$. Draw a generic picture to help you think.

2. Determine if the following limit exists. If it does, find its value. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}$

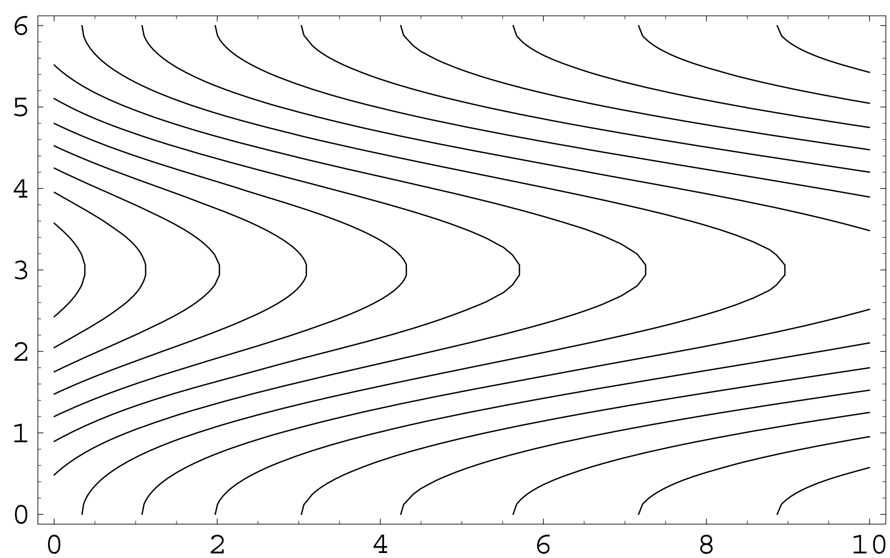
3. Consider the ellipse $x^2/4 + y^2/9 = 1$. Compute the curvature of the ellipse at the points where it (the curvature) is largest and smallest.

4. Let U be the subset of $\mathbf{R^2}$ where $y \neq 0$ and define $F : U \to \mathbf{R^2}$ by $F(x, y) = (x^2y, x/y)$. Let $P = (1, 2)$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$. Compute $D_{\mathbf{v}}F(P)$ in two ways.

   (a) Using the definition of $D_{\mathbf{v}}F(P)$.

   (b) Using the matrix for $DF(P)$.

5. Level curves of a function f are shown. Draw the five flow curves of ∇f that begin at $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, 4)$, and $(0, 5)$. Draw directly on this picture; do not use a separate sheet of paper. There will be very little partial credit on this problem.



6. Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $(x-2)^2 + (y-1)^2 + z^2 = 4$, and their locations. Do all calculus by hand. Clearly write down on your paper the algebraic equations to be solved. Use *Mathematica* to do the algebra. E-mail your *Mathematica* computation to me before you leave the exam.

7. Suppose $f : E^n \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are differentiable, and define $h : E^n \to \mathbf{R}$ by $h(X) = g(f(X))$. Prove both of the following.

   (a) $D_{\mathbf{v}} h(P) = g'(f(P)) D_{\mathbf{v}} f(P)$

   (b) $\nabla h(P) = g'(f(P)) \nabla f(P)$

   There are two approaches. Coordinate-free: Do (a) first, then (b) follows easily. With coorinates: Do (b) first, then (a) follows easily.

8. For each $\mathbf{F}$, determine (with explanation) if $\mathbf{F}$ is the gradient of some function f. If it is, find a function f by inspection such that $\nabla f = \mathbf{F}$. Note: Not being able to find a function f is not a sufficient explanation.

   (a) $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ (b) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

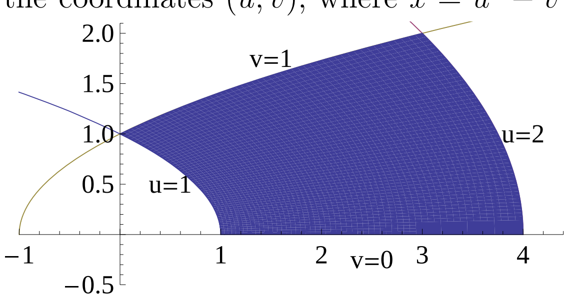
9. Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a C^2 vector field on $\mathbf{R^3}$. Show that $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

10. Given the following formulas for ∇f and $\operatorname{div} \mathbf{F}$ in polar coordinates, compute the formula for $\Delta f = \nabla^2 f$ (Laplacian) in polar coordinates.

$$\nabla f = (D_{\mathbf{u}_r} f)\mathbf{u}_r + (D_{\mathbf{u}_\theta} f)\mathbf{u}_\theta = \frac{\partial f}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta$$

$$\text{If } \mathbf{F} = P\mathbf{u}_r + Q\mathbf{u}_\theta, \quad \text{then} \quad \operatorname{div} \mathbf{F} = \frac{\partial P}{\partial r} + \frac{1}{r}P + \frac{1}{r}\frac{\partial Q}{\partial \theta}$$

11. Consider $\iint_R (x + y^2)\, dx\, dy$, where R is the region bounded by the curves $x = 1 - y^2$, $x = 4 - y^2/4$, $x = y^2 - 1$, and $y = 0$. Rewrite this integral in ready-to-evaluate form in the coordinates (u, v), where $x = u^2 - v^2$ and $y = uv$. Use *Mathematica* to evaluate.



12. Evaluate: $\int_0^2 \int_0^2 \int_0^2 \min(x, y, z)\, dx\, dy\, dz$

13. Let S be the solid region above the xy-plane inside the sphere of radius 3 centered at the origin. Express $\iiint_S z\, dV$ in both spherical and cylindrical coordinates. Evaluate one of them by hand.

14. Use Green's Theorem to compute the following path integral, where C is the boundary of the rectangle with vertices $(0,0)$, $(5,0)$, $(5,3)$, $(0,3)$, oriented counterclockwise.
$\int_C \big(\sin(e^x) - y^2\big)\,dx + \big(3x - \ln(1+y^2)\big)\,dy$

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

Have a good break!

Selected answers and hints.

1. $(-9/7, 8/7, 11/7)$ I can think of three different ways to do this.
2. The limit exists.
3. The maximum curvature is $3/4$.
4. $-\mathbf{i} - \frac{5}{4}\mathbf{j}$
5. Remember how ∇f is related to the level curves of f.
6. The maximum value is 8.63, which occurs at $(2.79, 2.07, 1.49)$.
8. One is a gradient, and one isn't.
9. This proof is in the book in Section 3.4.
11. $256/15$
12. 4 It is easy to get this answer for the wrong reason, so verify your method.
13. $81\pi/4$
14. 90