Math 225 **Final Exam** **Name:**

12 December 2005 200 Points

"Show enough work to justify your answers."

READ CAREFULLY: Do any **ten** problems. If you work on more than ten problems, you will get credit for the best ten. Note that some problems have multiple parts. If you use *Mathematica*, be very clear what you use it for, and show enough work that I can tell what you are thinking. Give exact answers when possible. Write integrals in ready-to-evaluate form before using *Mathematica*. If you have any question about what constitutes sufficient work, be sure to ask. Please do each problem on a separate page. Avoid writing anything in the upper left corner where the staple will go. (20 points each)

The topics are arranged roughly in the order we studied them.

1. Consider the plane determined by the points $A(0,2,0)$, $B(1,3,0)$, and $C(2,1,1)$.
   (a) Give parametric and implicit equations of the plane.
   (b) Let l be the line passing through the points $P(1,1,0)$ and $Q(2,0,-1)$. Find the point where l intersects the plane. Suggestion: Parameterize the line.

2. Suppose $\mathbf{v} : \mathbf{R} \to \mathbf{R^n}$ is a differentiable vector function.
   (a) Compute an expression for $\frac{d}{dt}\|\mathbf{v}(t)\|^2$ in terms of $\mathbf{v}(t)$ and $\mathbf{v}'(t)$. For full credit, your computation must be coordinate-free.
   (b) Use your answer in part a) to prove that $\mathbf{v}(t)$ has constant length if and only if $\mathbf{v}(t)$ and $\mathbf{v}'(t)$ are always perpendicular.

3. Consider the parameterization of the parabola $x = y^2$ given by $x = t^2$ and $y = t$. Compute the curvature as a function of t. Make a reasonably accurate and appropriately sized sketch of the parabola for $-2 \le t \le 2$. For $t = 0$ add the vectors $\mathbf{T}$ and $\mathbf{N}$ and the osculating circle to the sketch. Be sure everything is to scale.

4. Use the formula $\mathbf{a} = \dfrac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\dfrac{ds}{dt}\right)^2\mathbf{N}$ to prove $\kappa = \dfrac{\mathbf{a}\cdot\mathbf{v}_\perp}{\|\mathbf{v}\|^3}$ for a curve in $\mathbf{R^2}$.

5. (112) Let $f(x,y) = ye^{2x}\cos y$. Let P be the point $(0,\pi/4)$. Give a formula for the best linear approximation of f near P, and use it to approximate the value of $f(-.2, \pi/4 + .3)$.

6. Consider the surface $xyz = 1$. Show that the position vector of a point on the surface (the vector from the origin to the point) is never tangent to the surface. (Hints: What vector is normal to the surface? How can you tell if a vector is tangent to the surface?)

7. Let $f(x, y) = |x^2 - y^2|$. Determine, with proof, if f is differentiable at $(0, 0)$.

8. Consider the function $f(x, y) = x - \frac{1}{2}y^2$.

   (a) Make a reasonably accurate and appropriately sized sketch of the five level curves of f corresponding to the values -2, -1, 0, 1, 2. Label each curve with its value.

   (b) A large family of bugs is out for a walk on the xy-plane. Each bug walks in the direction that f increases most quickly. Draw arrows at several points indicating the direction a bug would go at that point.

9. (112) Find the global maximum and minimum values of $f(x, y) = x^2 - x + 2y^2$ on the closed unit disk $x^2 + y^2 \leq 1$.

10. (112) Let $f(x, y) = kx^2 + 2xy + ky^2$. This function has a critical point at $(0, 0)$. You do not need to verify this. For what values of k will f have a

    (a) Nondegenerate local minimum at $(0, 0)$?

    (b) Nondegenerate local maximum at $(0, 0)$?

    (c) Saddle point at $(0, 0)$?

11. (112) Write $\displaystyle\int_0^2 \int_{x^2}^{2x} (x - 3y^2)\, dy\, dx$ with the order of integration reversed. Evaluate one of the integrals by hand. You may leave the answer as a sum of fractions.

12. Consider the solid S in $\mathbf{R^3}$ that is above the xy-plane, inside the sphere of radius 2 centered at the origin, and inside the circular cone with vertex at the origin and axis along the z-axis making an angle of $60°$ with the xy-plane. (See picture on the board.)

    (a) Express $\iiint_S (x^2+y^2+z^2)^{3/2}\, dV$ in **both** spherical and cylindrical coordinates. Your integrals should be simplified and ready to evaluate.

    (b) Evaluate one of the integrals by hand.

13. Suppose that R is a region in $\mathbf{R^2}$ and that $f : R \to \mathbf{R}$ is differentiable. The surface area of the graph of f is given by $S = \iint_R \sqrt{1+||\nabla f||^2}\, dA$. Let $f(x,y) = 4 - x^2 - y^2$. Find the surface area of the portion of the graph of f that is above the xy-plane. For full credit, evaluate the integral by hand.

14. Suppose that R is a bounded region in $\mathbf{R^2}$ with piecewise continuously differentiable boundary.

    (a) Use Green's Theorem to explain why $\oint_C -y\, dx + x\, dy$ is two times the area of R, where C is the boundary of R oriented counterclockwise.

    (b) Use the path integral formula in part a) to compute the area of the elliptical disk $\dfrac{x^2}{9} + \dfrac{y^2}{4} \leq 1$. Evaluate the integral by hand.

15. Find the value of $\int_C (2y + \sqrt{9+x^3})\, dx + (5x + e^{\sin y})\, dy$, where C is the circle $x^2+y^2 = 9$ traversed clockwise. Do all computations by hand.

16. Consider the two vector fields.

$$\mathbf{F} = (y^2 - x^2 + 3)\mathbf{i} + (xy + \cos 2y)\mathbf{j} \qquad \mathbf{G} = (y^2 - 3x^2 + 2)\mathbf{i} + (2xy + \cos 3y)\mathbf{j}$$

    (a) Show that one of the vector fields is conservative and the other is not.

    (b) Find a potential function for the conservative field.

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

Have a good break!