Math 225 **Final Exam** **Name:**

15 December 2004 200 Points

READ CAREFULLY: Do any **ten** problems. If you work on more than ten problems, you will get credit for the best ten. Note that some problems have multiple parts. If you use *Mathematica*, be very clear what you use it for, and show enough work that I can tell what you are thinking. Give exact answers when possible. Write integrals in ready-to-evaluate form before using *Mathematica*. If you have any question about what constitutes sufficient work, be sure to ask. Please do each problem on a separate page. Avoid writing anything in the upper left corner where the staple will go. (20 points each)

1. Consider the plane determined by the points $A(5,0,0)$, $B(6,0,2)$, and $C(5,1,3)$.
   (a) Find an equation of the plane.
   (b) Let ℓ be the line passing through the points $P(1,1,0)$ and $Q(2,0,-2)$. Find the point where ℓ intersects the plane. Suggestion: Parameterize the line.

2. Suppose $\mathbf{v} : \mathbf{R} \to \mathbf{R}^n$ is a differentiable vector function.
   (a) Compute an expression for $\frac{d}{dt}\|\mathbf{v}(t)\|^2$ in terms of $\mathbf{v}(t)$ and $\mathbf{v}'(t)$. For full credit, your computation must be coordinate-free.
   (b) Use your answer in part a) to prove that $\mathbf{v}(t)$ has constant length if and only if $\mathbf{v}(t)$ and $\mathbf{v}'(t)$ are always perpendicular.

3. Consider the parameterization of the parabola $x = y^2$ given by $x = t^2$ and $y = t$. Compute the curvature as a function of t. Make a reasonably accurate and appropriately sized sketch of the parabola for $-2 \le t \le 2$. For $t = 0$ add the vectors $\mathbf{T}$ and $\mathbf{N}$ and the osculating circle to the sketch. Be sure everything is to scale.

4. Use the formula $\mathbf{a} = \dfrac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\dfrac{ds}{dt}\right)^2\mathbf{N}$ to prove $\kappa = \dfrac{\mathbf{a}\cdot\mathbf{v}_\perp}{\|\mathbf{v}\|^3}$ for a curve in $\mathbf{R}^2$.

5. Define $F : \mathbf{R}^2 \to \mathbf{R}^2$ by $F(u,v) = (u^2 - v^2, uv)$. Let P be the point $(u,v) = (3,-1)$, and let $\mathbf{w} = \mathbf{i} + 3\mathbf{j}$. Compute $D_\mathbf{w}F(P)$ in **both** of the following ways:
   (a) Using the definition of $D_\mathbf{w}F(P)$,
   (b) By computing the matrix for the linear map $DF(P)$ and using matrix multiplication.

6. Consider the surface $xyz = 1$. Show that the position vector of a point on the surface (the vector from the origin to the point) is never tangent to the surface. (Hints: What vector is normal to the surface? How can you tell if a vector is tangent to the surface?)

7. Let $f(x,y) = |x^2 - y^2|$. Determine, with proof, if f is differentiable at $(0,0)$.

8. Consider the function $f(x,y) = x + \frac{1}{2}y^2$.

   (a) Make a reasonably accurate and appropriately sized sketch of the five level curves of f corresponding to the values -2, -1, 0, 1, 2. Label each curve with its value.

   (b) A large family of bugs is out for a walk on the xy-plane. Each bug walks in the direction that f increases most quickly. Draw arrows at several points indicating the direction a bug would go at that point.

9. Prove that $x^2 - y^2 = 1$ is a smooth one-dimensional manifold. Explain why $x^2 - y^2 = 0$ is **not** a smooth one-dimensional manifold. Note: We don't have a theorem that implies when something is not a manifold.

10. Let $\mathbf{u}(\theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ and define $T : \mathbf{R}^2 \to \mathbf{R}^3$ by

$$\begin{aligned} T(\theta,\phi) &= (3+\cos\phi)\mathbf{u}(\theta) + \sin\phi\,\mathbf{k} \\ &= (3+\cos\phi)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) + \sin\phi\,\mathbf{k} \qquad (*) \\ &= \big((3+\cos\phi)\cos\theta, (3+\cos\phi)\sin\theta, \sin\phi\big). \end{aligned}$$

This is the parameterization of the torus $\mathbf{T}$ that we have used before. The variables θ and ϕ can be used as coordinates on the torus, just as latitude and longitude are used as coordinates on the earth.

   (a) Find formulas for the coordinate vector fields $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as linear combinations of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Show that they are orthogonal.

   (b) If $f : \mathbf{T} \to \mathbf{R}$ is differentiable, find the formula for ∇f as a linear combination of $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.

Suggestions: Use the formula marked $(*)$ as a starting point for your computations. Keep the factor $(3+\cos\phi)$ together, that is, don't distribute across it.

11. (112) Write $\displaystyle\int_0^2 \int_{x^2}^{2x} (x - 3y^2)\,dy\,dx$ with the order of integration reversed. Evaluate one of the integrals by hand.

12. Define $g : \mathbf{R}^3 \to \mathbf{R}$ by $g(x,y,z) = \min\{x,y,z\}$. Evaluate $\displaystyle\int_0^1 \int_0^1 \int_0^1 g(x,y,z)\,dx\,dy\,dz$.

13. Suppose that R is a region in $\mathbf{R}^2$ and that $f : R \to \mathbf{R}$ is differentiable. The surface area of the graph of f is given by $S = \iint_R \sqrt{1 + ||\nabla f||^2}\, dA$. Let $f(x, y) = 4 - x^2 - y^2$. Find the surface area of the portion of the graph of f that is above the xy-plane. For full credit, evaluate the integral by hand.

14. The surface $\rho = \sin\phi$ (spherical coordinates) is pictured. Note that any plane that contains the z-axis slices this surface in two circles. The circles are tangent to each other and to the z-axis at the origin. Find the volume of the solid contained by this surface.

15. Let C be the arch of $y = \sin x$ from 0 to π. Find the length and centroid of C.

16. Suppose that R is a bounded region in $\mathbf{R}^2$ with piecewise continuously differentiable boundary. The rotational moment of R about the origin is defined to be $\iint_R r^2\, dA$, where r is the distance to the origin (the r of polar coordinates).

    (a) Use Green's Theorem to show that $\oint_C -x^2 y\, dx + xy^2\, dy$ computes this polar moment, where C is the boundary of R oriented counter clockwise.

    (b) Use the path integral formula in part a) to compute the polar moment of the elliptical disk $\dfrac{x^2}{9} + \dfrac{y^2}{4} \le 1$.

17. Find the value of $\int_C (2y + \sqrt{9 + x^3})\, dx + (5x + e^{\sin y})\, dy$, where C is the circle $x^2 + y^2 = 9$ traversed clockwise. Do all computations by hand.

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.