Math 225 **Exam 2** **Name:**

10 November 2009 100 Points

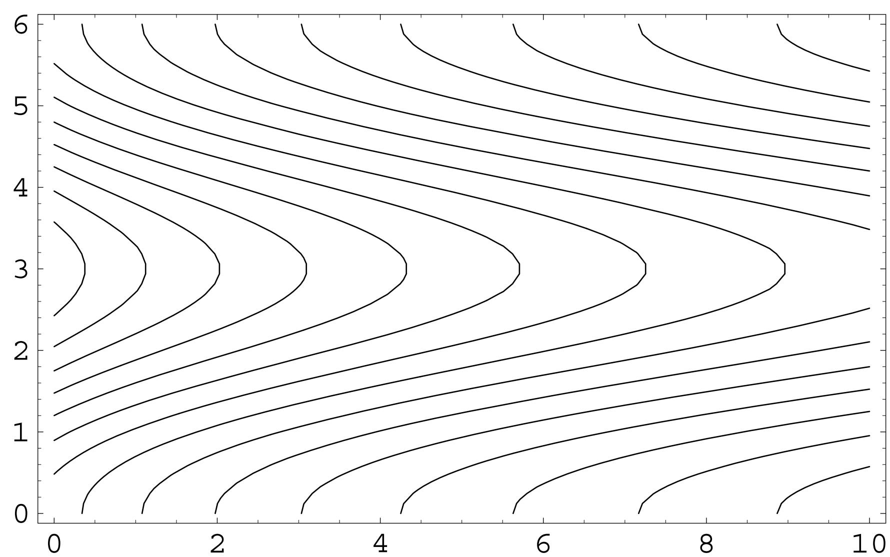
You may use *Mathematica* only as indicated in the Lagrange multipliers problem.

"Show enough work to justify your answers."

1. Let $\mathbf{F} = x^2z\mathbf{i} + xz^2\mathbf{j} + xy^2\mathbf{k}$. Compute $\operatorname{div}\mathbf{F}$ and $\operatorname{curl}\mathbf{F}$. (10 points)

READ CAREFULLY: Do **any six** of the remaining problems. They are worth 15 points each. If you work on more than six problems, you will get credit for the best six. **Please begin each problem on a new sheet of paper.** Put the problem number in the upper right corner of the page. Avoid writing in the upper left corner where the staple will go.

2. Level curves of a function f are shown. Draw the five flow curves of ∇f that begin at $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, and $(0,5)$. Draw directly on this picture; do not use a separate sheet of paper. There will be very little partial credit on this problem.



3. For each vector field $\mathbf{F}$, find a function f such that $\nabla f = \mathbf{F}$ or explain why there is no such function.
   (a) $\mathbf{F} = (x^2 - 2y)\mathbf{i} + (3x^2 + 2xy)\mathbf{j}$
   (b) $\mathbf{F} = (3x^2 + 2xy)\mathbf{i} + (x^2 - 2y)\mathbf{j}$

4. Let $f(x, y) = \cos(x - 2y)$. Compute the best quadratic approximation (2nd-order Taylor polynomial) of f at $(2, 1)$. Leave your answer in terms of $(x - 2)$ and $(y - 1)$.

5. Let $f(x, y, z) = x^2 + \sin(xy) + z^2$. Show that the origin is a critical point and determine if f has a local maximum, local minimum, or saddle point there.

6. Suppose $\gamma(t)$ is a curve and $\mathbf{F}$ is a vector field.

   (a) Describe in your own words and/or pictures what it means for $\gamma(t)$ to be a flow curve of $\mathbf{F}$. (5 points)

   (b) Determine if $\gamma(t) = (3\sin t, 2\cos t)$ is a flow curve of $\mathbf{F}(x, y) = \dfrac{3y}{2}\mathbf{i} - \dfrac{2x}{3}\mathbf{j}$.

7. Consider polar coordinates in the xy-plane.

   (a) Draw a representative sample of coordinate curves (at least five for each variable). Label each curve with its polar coordinate equation (some variable equals a constant). Draw with a reasonable degree of accuracy.

   (b) Compute formulas for the non-normalized coordinate vector fields $\mathbf{v}_r$ and $\mathbf{v}_\theta$.

   (c) Draw $\mathbf{v}_r$ and $\mathbf{v}_\theta$ on your coordinate curves at a minimum of four different locations. Do this in such a way as to illustrate that the vectors can change directions and magnitudes. Your vectors should be to scale.

8. In polar coordinates we have seen that the gradient of a function f and divergence of a vector field $\mathbf{F} = P\mathbf{u}_r + Q\mathbf{u}_\theta$ are given by

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{u}_\theta \qquad \text{and} \qquad \operatorname{div}\mathbf{F} = \frac{\partial P}{\partial r} + \frac{P}{r} + \frac{1}{r}\frac{\partial Q}{\partial \theta},$$

   where $\mathbf{u}_r$ and $\mathbf{u}_\theta$ are the unit coordinate vectors in this coordinate system.

   (a) Compute the Laplacian of a function, $\Delta f = \nabla^2 f$ in this coordinate system. (10 points)

   (b) Use your answer from part (a) to compute $\Delta f = \nabla^2 f$ for $f(r, \theta) = r^3 \sin(2\theta)$. (5 points)

9. Suppose that $z = f(x, y)$ is a differentiable function of $x = r\cos\theta$ and $y = r\sin\theta$. Show that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

10. Find the maximum and minimum values of $f(x, y, z) = x + 2y - 3z$ on the sphere $(x-2)^2 + (y-1)^2 + z^2 = 4$, and their locations. Do all calculus by hand. Clearly write down on your paper the full algebraic equations to be solved. Use *Mathematica* to do the algebra. Give exact answers and numerical approximations. There is a *Mathematica* template for this problem in N:/Math/Math225/Foote/Exam2.nb. E-mail your *Mathematica* computation to me before you leave the exam.

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

Selected answers and hints.

1. $\operatorname{div} \mathbf{F} = 2xz, \qquad \operatorname{curl} \mathbf{F} = (2xy - 2xz)\mathbf{i} - (y^2 - x^2)\mathbf{j} + z^2\mathbf{k}$

2. The flow curves of ∇f are ... to the level curves of f.

3. (a) Not a gradient because $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$. (b) It is the gradient of $f(x, y) = x^2y + x^3 - y^2$.

4. $f(x, y) \approx 1 + \frac{1}{2}(-(x-2)^2 + 4(x-2)(y-1) - 4(y-1)^2)$

5. Saddle. Note: You don't need to find all of the critical points to verifying that the origin is one.

6. (a) There's a nice description and picture of this in the book. (b) Yes, it is a flow curve. You do not need to solve the differential equations to determine this.

7. (a) Some of the coordinate curves are circles centered at the origin, some are rays coming from the origin.

   (b) $\mathbf{v}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \qquad \mathbf{v}_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}$

8. (a) $\nabla^2 f = \operatorname{div}(\nabla f) = f_{rr} + f_r/r + f_{\theta\theta}/r^2$

   (b) $8r\sin(2\theta)$

9.
   - Method I. From the Chain Rule, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$ and $\frac{1}{r}\frac{\partial z}{\partial r} = -\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta$. Square these and add them together.
   - Method II. With $z = f(x, y)$, the RHS is $\|\nabla f\|^2$ in Cartesian coordinates. The LHS is $\|\nabla f\|^2$ in polar coordinates (from the previous problem).

10. The equations to be solved are

$$1 = 2\lambda(x-2), \qquad 2 = 2\lambda(y-1), \qquad -3 = 2\lambda z, \qquad (x-2)^2 + (y-1)^2 + z^2 = 4.$$

These can be solved by hand (a bit messy—it would be easier if the sphere were centered at the origin):

$$(2 - \sqrt{2/7}, 1 - 2\sqrt{2/7}, 3\sqrt{2/7}) \qquad \text{and} \qquad (2 + \sqrt{2/7}, 1 + 2\sqrt{2/7}, -3\sqrt{2/7}).$$

The maximum value of f is $4 + 2\sqrt{14}$, which occurs at the first critical point. The minimum value of f is $4 - 2\sqrt{14}$, which occurs at the second critical point.