Math 225 **Exam 2** **Name:**

8 November 2007 100 Points

You may use *Mathematica* only as indicated in the Lagrange multipliers problem.

"Show enough work to justify your answers."

1. Find an equation of the plane tangent to $xz = (xy+3)^2$ at the point $(1, -1, 4)$. (7 points)

2. Consider the level curves of a function f as shown. Add ∇f to the picture at the indicated points P, Q, and R. Your vectors do not need to be to scale. (3 points)

READ CAREFULLY: Do **any six** of the remaining problems. They are worth 15 points each. If you work on more than six problems, you will get credit for the best six. **Please begin each problem on a new sheet of paper.** Put the problem number in the upper right corner of the page. Avoid writing in the upper left corner where the staple will go.

3. Let $f(x, y, z) = yx - xyz - z^2 - x^2 - y^2$. Show that the origin is a critical point and determine if f has a local maximum, local minimum, or saddle point there.

4. The surface $xyz = 1$ and the plane $x + 2y + 3z = 9$ intersect in the first octant (where the coordinates are all positive) in a loop. Since the loop is a closed and bounded set, there will be point on the loop closest to the origin, and another point farthest from the origin. Use Lagrange multipliers to find these points. Do all calculus by hand. You will need *Mathematica* to do the algebra. There is a template in Exam2.nb set up for this problem in the Math225/Foote folder for you to use. After you are done with *Mathematica*,

   (a) Record the locations of the closest and farthest points and their distances on your exam paper to two decimal places (be sure to only use points for which the coordinates are all positive),

   (b) Save the *Mathematica* file to your user area, and

   (c) Send the *Mathematica* file to me by e-mail (footer).

5. Not every vector field is a gradient.

   (a) Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. Find a function f such that $\nabla f = \mathbf{F}$.

   (b) Let $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$. Carefully explain why there is no function f such that $\nabla f = \mathbf{F}$.

6. If $f : \mathbf{R^2} \to \mathbf{R}$ is differentiable, compute the expression for ∇f in polar coordinates, that is, ∇f should be expressed as a linear combination of $\mathbf{u}_r$ and $\mathbf{u}_\theta$, and the coefficients should be expressed in terms of r, θ, and f. To do this you will need to first compute the coordinate vector fields $\mathbf{v}_r$ and $\mathbf{v}_\theta$.

7. Let $f(x, y) = x^3 - xy^2$. Compute the quadratic approximation (second-order Taylor polynomial) of f at $(2, -1)$. Leave your answer in terms of $(x-2)$ and $(y+1)$.

8. Derive either of the following formulas. If you work on both, you will get credit for the best one.

   (a) Suppose $f, g : \mathbf{R^2} \to \mathbf{R}$ are differentiable. Define $h : \mathbf{R^2} \to \mathbf{R}$ by $h(X) = f(X)g(X)$. Show that $\nabla h(X) = g(X)\nabla f(X) + f(X)\nabla g(X)$.

   (b) Suppose $f : \mathbf{R^2} \to \mathbf{R}$ and $G : \mathbf{R} \to \mathbf{R}$ are differentiable. Define $H : \mathbf{R^2} \to \mathbf{R}$ by $h(X) = G(f(X))$. Show that $\nabla H(X) = G'(f(X))\nabla f(X)$.

9. Suppose that temperature in the (x, y)-plane is given by $T(x, y) = 2x^2 + y^2$. A heat-seeking bug is walking in the plane. It always walks in the direction the temperature increases most quickly. Find the path of the bug if it starts at the point $(2, 1)$.

10. Suppose that $z = f(x + y, x - y)$ has continuous partial derivatives with respect to $u = x + y$ and $v = x - y$. Show that

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2$$

11. The basic iteration formula for the method of steepest ascent is written as

$$X_{n+1} = X_n + k\nabla f(X_n) \qquad \text{or} \qquad T(X) = X + k\nabla f(X),$$

where $k > 0$. Explain in your own words what the method involves and why it is a reasonable way to look for a local maximum. Make a copy of the figure in Problem 2 (or draw a similar figure of level curves) and use it to illustrate the process. You do not need to address the issue of what can go wrong if the value of k is not chosen carefully.

Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

Selected answers and hints.

1. $-8(x-1)+4(y+1)-(z-4)=0$

2.

3. Local maximum

4. For max/min problems involving distance, you can use the square of the distance function. In this case $f(x,y,z)=x^2+y^2+z^2$, since we are measuring distance from the origin. The constraint functions are $g_1(x,y,z)=xyz$ and $g_2(x,y,z)=x+2y+3z$. You may use Lagrange multipliers as $\nabla f=\lambda_1\nabla g_1+\lambda_2\nabla g_2$ or you may use $\det(\nabla f,\nabla g_1,\nabla g_2)=0$.

5. (a) $f(x,y)=xy$, (b) $Q_x\neq P_y$ for this vector field.

6. See 2008 exam for the formula.

7. $f(x,y)\approx 6+11(x-2)-4(y-1)+\frac{1}{2}(12(x-2)^2-4(x-2)(y+1)-4(y+1)^2)$

8. In both parts, you should write X as (x,y) and then compute partial derivatives to get the components of the gradients.

9. $y=x^2/4$.

10. From $z=f(u,v)=f(x+y,x-y)$ we have $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$; $\frac{\partial z}{\partial y}$ is similar.