Math 225 **Exam 2** **Name:**

14 November 2006 100 Points

You may use *Mathematica* only as indicated in the Lagrange multipliers problem.

"Show enough work to justify your answers."

1. Find an equation of the plane tangent to $z^2 + 1 = x^2 + 4y^2$ at the point $(1, -1, 2)$. (7 points)

2. Consider the level curves of a function f as shown. Add ∇f to the picture at the indicated points P, Q, and R. Your vectors do not need to be to scale. (3 points)

READ CAREFULLY: Do **any six** of the remaining problems. They are worth 15 points each. If you work on more than six problems, you will get credit for the best six. **Please begin each problem on a new sheet of paper.** Put the problem number in the upper right corner of the page. Avoid writing in the upper left corner where the staple will go.

3. Let $f(x, y, z) = x^2 + 2y^2 - y \sin z + z^2$. Show that the origin is a critical point and determine if f has a local maximum, local minimum, or saddle point there.

4. Find the maximum and minimum values of $f(x, y, z) = x - y + z$ and their locations on the curve of intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the paraboloid $z = x^2 + 2y^2$. Do all calculus by hand. You will need *Mathematica* to do the algebra. There is a template in Exam2.nb set up for this problem in the Math225/Foote folder for you to use. After you are done with *Mathematica*,

   (a) Record the maximum and minimum values of f and their locations on your exam paper to two decimal places,

   (b) Save the *Mathematica* file to your user area, and

   (c) Send the *Mathematica* file to me by e-mail (footer).

5. Let $f(x, y) = x^2 y$, $P = (2, -1)$, and $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}$.

   (a) Compute $D_{\mathbf{v}} f(P)$ using the definition of derivative along a vector.

   (b) Compute $D_{\mathbf{v}} f(P)$ using the the gradient of f.

6. Suppose that $f : \mathbf{R^2} \to \mathbf{R}$ or $f : \mathbf{R^3} \to \mathbf{R}$ (your choice), and that the second-order partial derivatives of f are continuous. Prove that $\text{curl}(\nabla f) = 0$. Note: You may not choose a specific function.

7. Not every vector field is a gradient.

   (a) Let $\mathbf{F} = x^2\mathbf{i} + (1/y)\mathbf{j}$ on the part of $\mathbf{R^2}$ where $y > 0$. Find a function f such that $\nabla f = \mathbf{F}$.

   (b) Let $\mathbf{F} = (1/y)\mathbf{i} + x^2\mathbf{j}$ on the part of $\mathbf{R^2}$ where $y > 0$. Explain why there is no function f such that $\nabla f = \mathbf{F}$.

8. If $f : \mathbf{R^2} \to \mathbf{R}$ is differentiable, compute the expression for ∇f in polar coordinates, that is, ∇f should be expressed as a linear combination of $\mathbf{u}_r$ and $\mathbf{u}_\theta$, and the coefficients should be expressed in terms of r, θ, and f. To do this you will need to first compute the coordinate vector fields $\mathbf{v}_r$ and $\mathbf{v}_\theta$.

9. Let $f(x, y) = \ln(x^2 + y^2)$. Show that f is harmonic in two ways.

   (a) Compute $\nabla^2 f = \Delta f$ in Cartesian coordinates.

   (b) Express f in polar coordinates and use the Laplacian in polar coordinates:

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

10. The spherical Laplacian operator is given by

$$\Delta_S f = \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2} + \cot\phi \frac{\partial f}{\partial \phi}.$$

    Show that $f(\theta, \phi) = \cos\phi$ is a spherical harmonic and determine its eigenvalue.

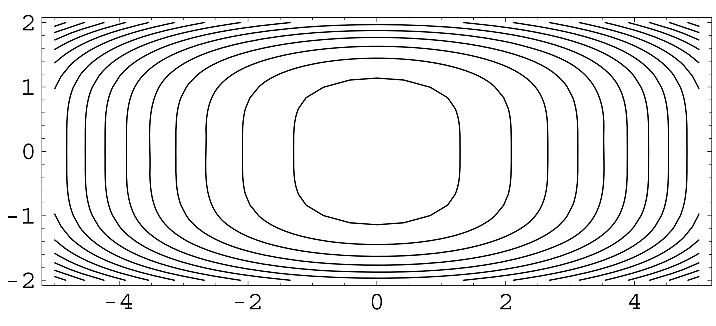
11. Suppose $X(t)$ is a curve and $\mathbf{F}$ is a vector field.

    (a) Describe in your own words and/or pictures what it means for $X(t)$ to be a flow curve of $\mathbf{F}$. (5 points)

    (b) Determine if $X(t) = (t^2, 1/t)$ is a flow curve of $\mathbf{F}(x, y) = 2xy\mathbf{i} - y^2\mathbf{j}$. (10 points)

12. Steepest Descent/Ascent. Let $f(x, y) = -x^2 - y^4$. Note that this takes its maximum at $(0, 0)$. Perform two steps of the Steepest Descent/Ascent algorithm starting at $(4, -1)$ using $k = 1/4$. Plot $(4, -1)$ and the two additional points on the picture. Also draw the displacement vectors that go from each approximation to the next. The iteration formula is written as

$$X_{n+1} = X_n + k\nabla f(X_n) \qquad \text{or} \qquad T(X) = X + k\nabla f(X).$$



Please put your pages in order before stapling and be sure the problem numbers are in the upper right corners.

Hints and selected answers.

1. $2(x-1) - 8(y+1) - 4(z-2) = 0$

3. The origin is a local minimum.

4. The maximum value is 3.22 occurring at $(.89, -.66, 1.66)$.

5. 20

6. The proof for $f : \mathbf{R^3} \to \mathbf{R}$ is in the text on page 218.

7. (b) If you compute curl $\mathbf{F}$, you don't get 0, and so $\mathbf{F}$ cannot be a gradient by the result in Problem 6.

10. The eigenvalue is -2.

11. (b) X is a flow curve of $\mathbf{F}$.

12. After two iterations you wind up at $(1, 0)$.