Math 225 **Exam 1** **Name:**

8 October 2009 100 Points

Note: You may use *Mathematica* to help you think in any problem, but you may not use it in any solution except as indicated.

"Show enough work to justify your answers."

READ CAREFULLY! This exam has **two** parts. In Part I you are are to do **all** four problems. In Part II you have a choice of problems.

Part I. Do **all** four problems in this part. Do your work on the exam paper. (10 points each)

1. Find an equation for the plane tangent to the graph of $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 11$ at the point $(-3, 2, 3)$.

2. Find a non-parametric equation for the plane that passes through the points $A(2, 0, 1)$, $B(0, 1, 3)$, and $C(1, 1, 1)$.

3. Find the distance between the point $P(5, 4, -3)$ and the plane $2x - y - 2z = 4$.

4. Let $f(x, y) = x^2/y$.

   (a) Compute the directional derivative of f at $(3, 1)$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$. (7 points)

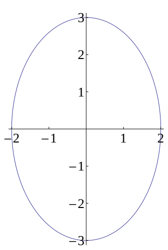
   (b) In which direction from $(3, 1)$ does f *decrease* most quickly? (Your answer should be a vector.) (3 points)

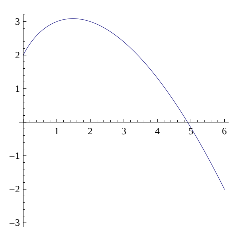
READ CAREFULLY! Do **any four** of the remaining problems. If you work on more than four, you will get credit for the best four. Please do each problem on a separate sheet of paper. Put the problem number in the **upper right corner** and avoid writing anything in the upper left corner where the staple will go. (15 points each)

5. The ellipse $9x^2 + 4y^2 = 36$ is shown.

   (a) Compute the curvature and radius of curvature at the point $P = (0, 3)$, assuming the ellipse is traversed counterclockwise. (10 points)

   (b) Plot the center of the osculating circle that is tangent at P and draw the circle. Give the coordinates of the center of the circle. Draw directly on the picture below. *Be sure your drawing is to scale.* (5 points)





6. Consider the pictured parameterized curve $\gamma(t) = (t^2 + t, -t^3 + 2t + 2)$ for $0 \leq t \leq 2$.

   (a) Compute the vectors $\mathbf{v}$, $\mathbf{a}$, $\mathbf{T}$, and $\mathbf{N}$ at $t = 1$. (8 points)

   (b) Add these vectors to the picture with their tails at $\gamma(1)$. Draw directly on the picture above. *Label your vectors and be sure they are to scale.* (7 points)

7. Let $\gamma(t) = (t^2 + t, -t^3 + 2t + 2)$ for $0 \leq t \leq 2$ (the same curve as in the previous problem).

   (a) Let $L(t)$ be the best affine approximation to $\gamma(t)$ at $t = 1$. Give the formula for $L(t)$. (6 points)

   (b) Write a simplified, ready-to-evaluate integral that gives the length of the curve. Use *Mathematica* to evaluate it. (You need to use NIntegrate. Syntax: **NIntegrate**[f[t],{t,a,b}] gives a numerical approximation to $\int_a^b f(t)\,dt$.) (9 points)

8. Suppose that the temperature in the (x, y)-plane is given by a function T such that $\nabla T = xy\mathbf{i} - \dfrac{x^2}{y}\mathbf{j}$. Suppose that an insect is moving in the plane according to the parameterized curve $\gamma(t) = (t^2, t^3)$. Determine the rate at which the insect feels the temperature changing at $t = 2$.

9. Do any **one** of the following proofs. If you work on more than one, you will get credit for the best one.

   (a) Prove the formula $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$.

   (b) Suppose $\mathbf{v}(t)$ is a differentiable function of $t \in \mathbf{R}$. Prove that $\mathbf{v}(t)$ has constant length if and only if $\mathbf{v}(t)$ and $\mathbf{v}'(t)$ are perpendicular for all t.

10. Let $f(x, y) = xy^2$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$, and $P(4, 1)$. Compute $D_{\mathbf{v}}f(P)$ in two ways.

    (a) Using the definition of $D_{\mathbf{v}}f(P)$. (9 points)

    (b) Using the gradient of f. (6 points)

11. Suppose $f : \mathbf{R^3} \to \mathbf{R}$.

    (a) Define carefully what is meant by a level surface of f. (7 points)

    (b) If $f(x, y, z) = x^2 + y^2 - z^2$, find an equation of the level surface of f that passes through $(0, 1, -2)$. (8 points)

Selected answers and hints.

1. $-18(x+3)+3(y-2)+2(z-3)=0$
2. A normal vector is $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$.
3. $8/3$
4. (a) $-18/5$
   (b) $-2\mathbf{i}+3\mathbf{j}$
5. (b) Center $(0,5/3)$, radius $4/3$
6. $\mathbf{N}=\frac{1}{\sqrt{10}}\mathbf{i}+\frac{3}{\sqrt{10}}\mathbf{j}$. Note that $\mathbf{N}$ points to the outside of the curve.
7. (a) $L(t)=(2,3)+(t-1)(3\mathbf{i}-\mathbf{j})=\big(2+3(t-1),3-(t-1)\big)$
   (b) 8.99501
8. 104. This is a chain rule problem. You should not try to determine a formula for T.
9. The other proof you should know is the proof of the curvature formula $\kappa=\frac{\mathbf{a}\cdot\mathbf{v}_\perp}{||\mathbf{v}||^3}$ starting from the formula for acceleration in part (a).
10. -5
11. $x^2+y^2-z^2=-3$