Math 225 **Exam 1** **Name:**

4 October 2007 100 Points

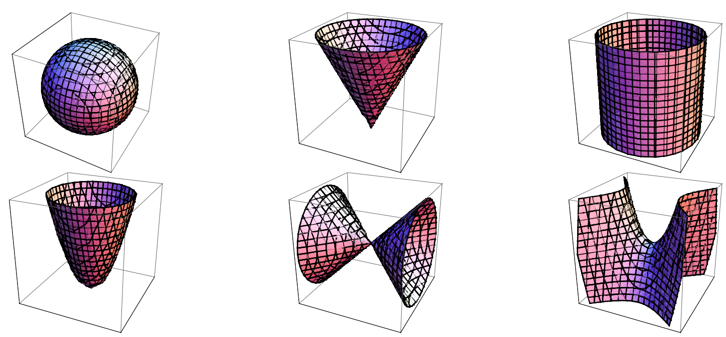
No *Mathematica*. You may use a calculator to do arithmetic, but exact answers are expected.

"Show enough work to justify your answers."

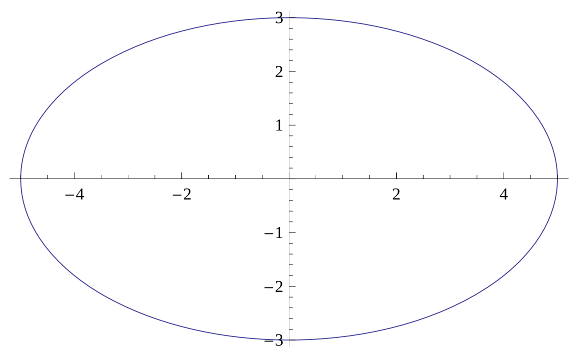
1. Match each graph with its equation. Put the letter of the graph in the blank next to the appropriate equation. Remember that you can determine some features of a surface by setting a variable in its equation to a constant. (10 points)

$z = x^2 + y^2$ ______ $z = y^2 - x^2$ ______ $y^2 = x^2 + z^2$ ______

$\rho = 1$ ______ $\phi = \pi/6$ ______ $r = 1$ ______



READ CAREFULLY! Do **any six** of the remaining problems. If you work on more than six, you will get credit for the best six. Please do each problem on a separate sheet of paper. Put the problem number in the **upper right corner** and avoid writing anything in the upper left corner where the staple will go. (15 points each)



2. The ellipse $9x^2 + 25y^2 = 225$ is shown.

   (a) Compute the curvature at the point $P = (5, 0)$, assuming the ellipse is traversed counterclockwise.

   (b) Draw the vectors $\mathbf{T}$ and $\mathbf{N}$ at P. Be sure to draw them to scale. Draw directly on the picture above.

   (c) Plot the center of the osculating circle that is tangent at P and draw the circle. You do not need to give the coordinates of the center of the circle, but indicate how far it is from the curve.

3. Do any **one** of the following proofs. If you work on more than one, you will get credit for the best one.

   (a) Prove the formula $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$.

   (b) Starting with the formula in the previous part, state and prove the formula for curvature.

   (c) Suppose $\mathbf{v}(t)$ is a differentiable function of $t \in \mathbf{R}$. Prove that $\mathbf{v}(t)$ has constant length if and only if $\mathbf{v}(t)$ and $\mathbf{v}'(t)$ are perpendicular for all t.

4. Explain why the following equations represent parallel (non-intersecting) planes, and find the distance between the the planes.

$$3x + y - 2z = 1 \qquad\qquad 3x + y - 2z = 10$$

5. Let S be the sphere $x^2 + (y-2)^2 + (z+3)^2 = 4$. For (x, y, z) in $\mathbf{R^3}$, let $f(x, y, z)$ be the distance from (x, y, z) to S, and let $F(x, y, z)$ be the point on S closest to (x, y, z). Give formulas in coordinates for f and F.

6. Define $F : \mathbf{R^3} \to \mathbf{R^2}$ by $F(x, y, z) = (xy, y/z)$. Let $P = (2, 1, 3)$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. Compute $D_{\mathbf{v}}F(P)$ using any method you wish.

7. Consider the graph of $xy = 1$ in the third quadrant. Parameterize this curve by the slope of the position vector, that is, find formulas for x and y in terms of t so that (x, y) is on the curve and t is the slope of the vector $x\mathbf{i} + y\mathbf{j}$.

8. Let $f(x, y) = \sqrt{2}\, e^{2x} \cos y$. Let P be the point $(0, \pi/4)$.

   (a) Compute $\nabla f(P)$.

   (b) Use the best linear approximation of f near P to approximate the value of $f(-.2, \pi/4 + .3)$.

9. The map $(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$ defines polar coordinates in the (x, y)-plane. Let $P = F(2, \pi/3)$.

   (a) Compute P and the coordinate vectors $\frac{\partial F}{\partial r}$ and $\frac{\partial F}{\partial \theta}$ at P.

   (b) Plot the point P in the (x, y)-plane. Draw the coordinate vectors at P and label them. Draw accurately, to scale, and not too small.

10. Find an equation of the plane tangent to the surface $2xz + yz + 10 = x^2y$ at the point $(1, -5, 5)$.

- Put problem numbers in the **upper right corner** of each page.
- Put your pages in order.

Selected answers and hints.

2. (a) $\kappa = 5/9$

4. The distance is $9/\sqrt{14}$. There are two ways to find this: use scalar projection or parameterize some line perpendicular to the planes.

6. $-3\mathbf{i} - \mathbf{j}$. There are two ways to do this: use the definition of $D_{\mathbf{v}}F(P)$ or compute the matrix for $DF(P)$.

7. $(x, y) = (-1/\sqrt{t}, -\sqrt{t})$

8. .3

9. $P = (1, \sqrt{3})$, $\frac{\partial F}{\partial r}(P) = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$, $\frac{\partial F}{\partial \theta}(P) = -\sqrt{3}\mathbf{i} + \mathbf{j}$. Noting that the polar coordinates of P are $(2, \pi/3)$ and that $||\frac{\partial F}{\partial r}(P)|| = 1$ and $||\frac{\partial F}{\partial r}(P)|| = 2$ helps to plot these.

10. A very common error is to forget the "$= 0$" that makes it an equation.