Math 225 **Exam 1** **Name:**

3 October 2006 100 Points

You may use *Mathematica* to help you think, but you may not use it in any solution.

"Show enough work to justify your answers."

1. Let L_1 be the line joining $P(2,3)$ and $Q(-1,2)$. Let L_2 be the line joining $R(1,1)$ and $S(0,-2)$. Let θ be either of the angles between the lines. Find the exact value of $\cos\theta$. Use a calculator or *Mathematica* to give an approximation for θ. Do your work in the space below. (10 points)

READ CAREFULLY! Do **any six** of the remaining problems. If you work on more than six, you will get credit for the best six. Please do each problem on a separate sheet of paper. Put the problem number in the **upper right corner** and avoid writing anything in the upper left corner where the staple will go. (15 points each)

2. Prove the formula $\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$.

3. Starting with the formula in the previous problem, state and prove the formula for curvature.

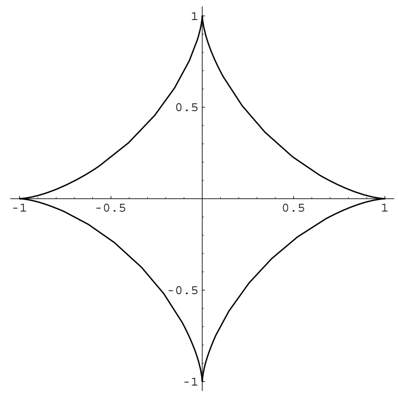
4. Suppose $\mathbf{v}(t)$ is a differentiable function of $t \in \mathbf{R}$. Prove that $\mathbf{v}(t)$ has constant length if and only if $\mathbf{v}(t)$ and $\mathbf{v}'(t)$ are perpendicular for all t.

5. Consider points $P(0,2,-1)$ and $Q(1,1,2)$ in $\mathbf{R^3}$ and the vector $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. Let L be the line passing through P parallel to $\mathbf{v}$. Find parametric and non-parametric equations for the plane containing L and Q.

6. Define $F : \mathbf{R^3} \to \mathbf{R}$ by $F(x,y,z) = x^2 + (y-2)^2 + (z+3)^2$. Give the equation of the level surface of F that passes through the point $(1,1,1)$. Describe the surface in words as precisely as you can (it is a very simple description).

7. Define $F : \mathbf{R^3} \to \mathbf{R^2}$ by $F(x, y, z) = (xy, y/z)$. Compute the 2×3 matrix for $DF(P)$, where P is the point $(2, 1, 3)$.

8. Determine, with justification, if $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ exists. If the limit exists, give its value.



9. The pictured curve is $x^{2/3} + y^{2/3} = 1$. A parameterization of this curve is $\gamma(t) = (\cos^3 t, \sin^3 t)$.

   (a) Plot the point $\gamma(\pi/4)$ on the curve and draw the vectors $\mathbf{T}$ and $\mathbf{N}$ at this point. Be sure to draw the vectors to scale. Draw directly on the picture above.

   (b) Compute the curvature at the point where $t = \pi/4$. For your convenience, here are $\gamma'(t)$ and $\gamma''(t)$.

$$\gamma'(t) = 3 \sin t \cos t(-\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$$

$$\gamma''(t) = 3 \sin t \cos t(\sin t\,\mathbf{i} + \cos t\,\mathbf{j}) + 3(\cos^4 t - \sin^4 t)(-\cos t\,\mathbf{i} + \sin t\,\mathbf{j})$$

   (c) Plot the center of the osculating circle for $t = \pi/4$ and draw at least part of the circle. You do not need to give the coordinates of the center of the circle, but indicate how far it is from the curve and in which direction.

10. Consider the curve $x^{2/3} + y^{2/3} = 1$ in the previous problem.

   (a) Show that $\gamma(t) = (\cos^3 t, \sin^3 t)$ is a parameterization of this curve.

   (b) Give the parameterization $\ell(t)$ of the line tangent to $\gamma(t)$ at $t = \pi/4$ with the property that $\ell(\pi/4) = \gamma(\pi/4)$.

   (c) Compute the length of one-fourth of γ. (You may use information in the previous problem.)

Selected answers and hints.

1. You know how to find the angle between two vectors, so you need to find appropriate vectors to use. The angle is approximately 53° or 127°, depending on which way you measure it.

5. Be sure not to use the position vectors of P and Q in your solution. They have no meaning for the geometry of the problem. There are several correct ways to write the equations. One form for the parametric equation is

$$(x, y, z) = (-s + t, 2 + 3x - t, -1 + s + 3t).$$

   If you multiply out the non-parametric equation and reduce it, you get $5x+2y-z = 5$. One way to check your equations is to see if they are satisfied by three points you know are on the plane, for example, P, Q, and $P + \mathbf{v}$.

6. It is a sphere of radius $3\sqrt{2}$ centered at $(0, 2, -3)$.

7. $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1/3 & -1/9 \end{pmatrix}$

8. Show that $\left| \dfrac{x^2 y}{x^2 + y^2} \right| \le |y|$.

9. (a) Locate $\gamma(0)$ and $\gamma(\pi/2)$ to determine which direction the curve is going. Be sure to draw $\mathbf{T}$ and $\mathbf{N}$ the right length.
   (b) $\kappa = -2/3$.

10. (b) $\ell(t) = \left(\frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}(t - \pi/4), \frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}(t - \pi/4) \right)$
    (c) 3/2