THE PERP OPERATOR ON $\mathbb{R}^2$

MATH 225

Definition. For any vector $\mathbf{v}$ in $\mathbb{R}^2$, let $\mathbf{v}_\perp$ denote the vector obtained by rotating $\mathbf{v}$ counter clockwise 90°. This defines an operation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

1. Basic properties

$$(\mathbf{v}_\perp)_\perp = -\mathbf{v}$$

Perp preserves norms and dot products: $\quad \|\mathbf{v}_\perp\| = \|\mathbf{v}\|, \quad (\mathbf{v}_\perp \cdot \mathbf{w}_\perp) = \mathbf{v} \cdot \mathbf{w}$

Perp is linear: $\quad (a\mathbf{v})_\perp = a\mathbf{v}_\perp, \quad (\mathbf{v}+\mathbf{w})_\perp = \mathbf{v}_\perp + \mathbf{w}_\perp$

$|\mathbf{v}_\perp \cdot \mathbf{w}|$ is the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$

$$(\mathbf{v} \cdot \mathbf{w})^2 + (\mathbf{v}_\perp \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$$

$\mathbf{v}_\perp \cdot \mathbf{w} > 0$ iff $\mathbf{v}$ and $\mathbf{w}$ are positively oriented
$\mathbf{v}_\perp \cdot \mathbf{w} < 0$ iff $\mathbf{v}$ and $\mathbf{w}$ are negatively oriented
$\mathbf{v}_\perp \cdot \mathbf{w} = 0$ iff $\mathbf{v}$ and $\mathbf{w}$ are dependent

$$\mathbf{v}_\perp \cdot \mathbf{w} = -\mathbf{v} \cdot \mathbf{w}_\perp$$

$$\mathbf{i}_\perp = \mathbf{j} \qquad \text{and} \qquad \mathbf{j}_\perp = -\mathbf{i}$$

Since the operation $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $P(\mathbf{v}) = \mathbf{v}_\perp$ is linear, it follows from what it does to $\mathbf{i}$ and $\mathbf{j}$ that its matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and that

$$(a\mathbf{i} + b\mathbf{j})_\perp = -b\mathbf{i} + a\mathbf{j} = \det \begin{pmatrix} a & b \\ \mathbf{i} & \mathbf{j} \end{pmatrix}$$

2. Applications

The formulas in these applications are not meant to be memorized. Instead, you should focus on the use of the perp operation so that you can use it in other situations.

Finding coefficients in a linear combination. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis of $\mathbb{R}^2$ and that $\mathbf{w}$ is some vector in $\mathbb{R}^2$. Then

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$$

for some unique scalars a_1 and a_2. The perp operator can be used to write formulas for the coefficients as easily as if $\mathbf{v}_1$ and $\mathbf{v}_2$ were orthogonal. To solve for a_1, dot both sides of the equation with $\mathbf{v}_{2\perp}$ so that the $\mathbf{v}_2$ term drops out:

$$\mathbf{w}\cdot\mathbf{v}_{2\perp} = a_1\mathbf{v}_1\cdot\mathbf{v}_{2\perp} + a_2\mathbf{v}_2\cdot\mathbf{v}_{2\perp} = a_1\mathbf{v}_1\cdot\mathbf{v}_{2\perp}.$$

Similarly,

$$\mathbf{w}\cdot\mathbf{v}_{1\perp} = a_1\mathbf{v}_1\cdot\mathbf{v}_{1\perp} + a_2\mathbf{v}_2\cdot\mathbf{v}_{1\perp} = a_2\mathbf{v}_2\cdot\mathbf{v}_{1\perp} = -a_2\mathbf{v}_1\cdot\mathbf{v}_{2\perp}.$$

Note that $\mathbf{v}_1\cdot\mathbf{v}_{2\perp} \neq 0$ since $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent. Thus the coefficients are

$$a_1 = \frac{\mathbf{w}\cdot\mathbf{v}_{2\perp}}{\mathbf{v}_1\cdot\mathbf{v}_{2\perp}} \qquad \text{and} \qquad a_2 = -\frac{\mathbf{w}\cdot\mathbf{v}_{1\perp}}{\mathbf{v}_1\cdot\mathbf{v}_{2\perp}}.$$

Finding the intersection of two lines. Suppose that two lines in $\mathbb{R}^2$ are given parametrically as

$$X = P + t\mathbf{v} \qquad \text{and} \qquad X = Q + t\mathbf{w}.$$

Assuming that the lines are not parallel, we would like a formula for their point of intersection. The point will satisfy

$$P + t\mathbf{v} = Q + s\mathbf{w}$$

for some scalars t and s. (Note: There might not be any solution of the equation $P + t\mathbf{v} = Q + t\mathbf{w}$. Thinking of the parameterized lines as describing the motion of two particles as functions of time, even if the lines intersect, the particles need not go through the point of intersection at the same time. The use of different parameters allows the possibility that the particles go through the point of intersection at different times.)

We need to find either t or s. (Note: We only need to find one of them. If we find s, then we simply plug its value into $Q + s\mathbf{w}$, which gives the desired point.) If we subtract Q from both sides we get

$$(P - Q) + t\mathbf{v} = s\mathbf{w}.$$

This is now an equation of vectors (the previous one is an equation of points). Dotting both sides by $\mathbf{v}_\perp$ we get $(P-Q)\cdot\mathbf{v}_\perp = s\mathbf{w}\cdot\mathbf{v}_\perp$. Since the lines are not parallel, neither are $\mathbf{v}$ and $\mathbf{w}$, and so $\mathbf{w}\cdot\mathbf{v}_\perp \neq 0$. Thus $s = \frac{(P-Q)\cdot\mathbf{v}_\perp}{\mathbf{w}\cdot\mathbf{v}_\perp}$ and so the point of intersection is

$$Q + s\mathbf{w} = Q + \frac{(P-Q)\cdot\mathbf{v}_\perp}{\mathbf{w}\cdot\mathbf{v}_\perp}\mathbf{w}.$$

Problems

(1) Let $\mathbf{v}_1 = 2\mathbf{i}+3\mathbf{j}$, $\mathbf{v}_2 = 5\mathbf{i}-4\mathbf{j}$, and $\mathbf{w} = 3\mathbf{i}+2\mathbf{j}$, and write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Do this in two ways: a) Do it the way you would have in Math 223; b) Use formulas in Section 2 for the coefficients.

(2) Consider the line passing through $P(2,4)$ parallel to $\mathbf{v} = 3\mathbf{i}-2\mathbf{j}$ and the line passing through $Q(-3,4)$ parallel to $\mathbf{w} = \mathbf{i}+4\mathbf{j}$. Find the point of intersection of these two lines. Do this in three ways: a) Write the lines in level curve form, $ax+by=c$, and use high school algebra; b) Write the equation $P+t\mathbf{v} = Q+s\mathbf{w}$ out in coordinates, getting two equations in the unknowns t and s; c) Use the formula for s in Section 2.

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