When is a Vector Field the Gradient of a Function?
Insert for Colley's text, Sections 3.3 and 3.4.

This handout is related to "Gradient Fields and Potentials" in Section 3.3 and "Two Vector-analytic Results" in Section 3.4.

Suppose $D \subset E^n$ and $f : D \to \mathbb{R}$ is differentiable. Then $\mathbf{F} = \nabla f$ is a vector field on D. We want to address the reverse question: if $\mathbf{F}$ is a vector field on D, is it the gradient of some function? More specifically, given $\mathbf{F} : D \to V^n$ (a vector field on D),

1. Does there exist a differentiable function $f : D \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$?
2. If such a function exists, how can you find one?

This is partially answered by the following theorem, stated here in $\mathbb{R}^2$.

Theorem 1. Suppose $D \subset \mathbb{R}^2$ and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on D with C^1 coefficients. If $\mathbf{F}$ is the gradient of a function, then $P_y = Q_x$, where the subscripts denote partial derivatives.

(Note: A function is C^1 if it is differentiable and its partial derivatives are continuous.)

Proof: Suppose $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$. Then $P = f_x$ and $Q = f_y$. It follows that f satisfies the hypothesis for the theorem on equality of mixed partial derivatives. We then have

$$P_y = (f_x)_y = f_{xy} = f_{yx} = (f_y)_x = Q_x.$$

This theorem makes it easy to tell when a vector field is **not** a gradient: if $P_y \neq Q_x$, then $\mathbf{F}$ is not a gradient. For example, consider $\mathbf{F} = y^2\mathbf{i} + x\mathbf{j}$. We have $P = y^2$ and $Q = x$. Since $P_y = 2y$ and $Q_x = 1$, and so $P_y \neq Q_x$, it follows that $\mathbf{F}$ is not the gradient of any function.

Here is the theorem in $\mathbb{R}^3$.

Theorem 2. Suppose $D \subset \mathbb{R}^3$ and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on D with C^1 coefficients. If $\mathbf{F}$ is the gradient of a function, then

$$P_y = Q_x, \qquad P_z = R_x, \quad \text{and} \quad Q_z = R_y.$$

Note that the coefficients of the curl of a vector field consist precisely of the differences of these partial derivatives. Thus, a convenient way to state both theorems is the following. (This is Theorem 4.3 in Section 2.4, except there it is stated only in $\mathbb{R}^3$.)

Theorem 3. Suppose $D \subset \mathbb{R}^n$ with $n = 2$ or $n = 3$ and $\mathbf{F}$ is a C^1 vector field on D. If $\mathbf{F}$ is the gradient of a function, then $\operatorname{curl} \mathbf{F} = 0$.

So far we have a condition that says when a vector field is **not** a gradient. The converse of Theorem 1 is the following: Given vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ on D with C^1 coefficients, if $P_y = Q_x$, then $\mathbf{F}$ is the gradient of some function. Unfortunately, this is not quite true. The additional hypothesis is a geometric condition on D. The most general condition (called simple connectivity) is technical, and we will see it in Chapter 6, but here is a version that is adequate for our current purposes.

Theorem 4. Suppose $D \subset \mathbb{R}^2$ is a disk or the interior of a rectangle, and that $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}$ is a vector field on D with C^1 coefficients. If $P_y = Q_x$, then $\mathbf{F}$ is the gradient of some function.

The proof of this (along with the more general condition on D) is technical, and is given in Chapter 6.

For our purposes, the condition $P_y = Q_x$ can be interpreted as $\mathbf{F}$ is "probably" a gradient. For example, consider $\mathbf{F} = y\mathbf{i}+x\mathbf{j}$. We have $P = y$ and $Q = x$. Since $P_y = 1 = Q_x$, we expect $\mathbf{F}$ to be the gradient of some function. Indeed, if $f(x, y) = xy$, then $\nabla f = \mathbf{F}$. In simple cases, such as this one, the needed function can be found by "guess and check." In more complicated examples, there is a procedure, which is outlined in "Finding Scalar Potentials" in Section 6.3.

Here is a strengthened converse for Theorem 2.

Theorem 5. Suppose $D \subset \mathbb{R}^3$ is a solid ball or the interior of a box, and that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on D with C^1 coefficients. If

$$P_y = Q_x, \qquad P_z = R_x, \quad \text{and} \quad Q_z = R_y,$$

then $\mathbf{F}$ is the gradient of some function.

Assignment:

- Read "Finding Scalar Potentials" in Section 6.3.
- Problems, Section 6.3, #3–18. Note: When referring to a vector field, "conservative," "has a potential," and "is a gradient" mean the same thing.

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Fall 2009, last update Fall 2014