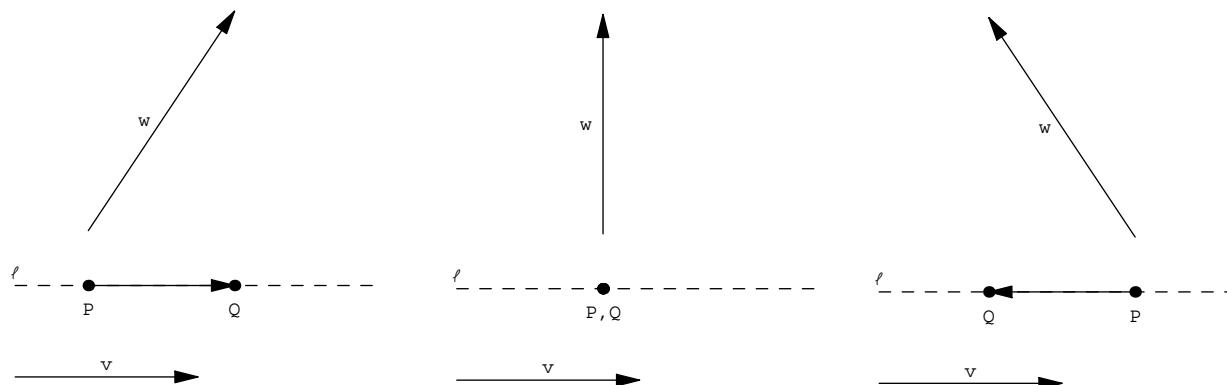


TWO TYPES OF PROJECTION

MATH 225

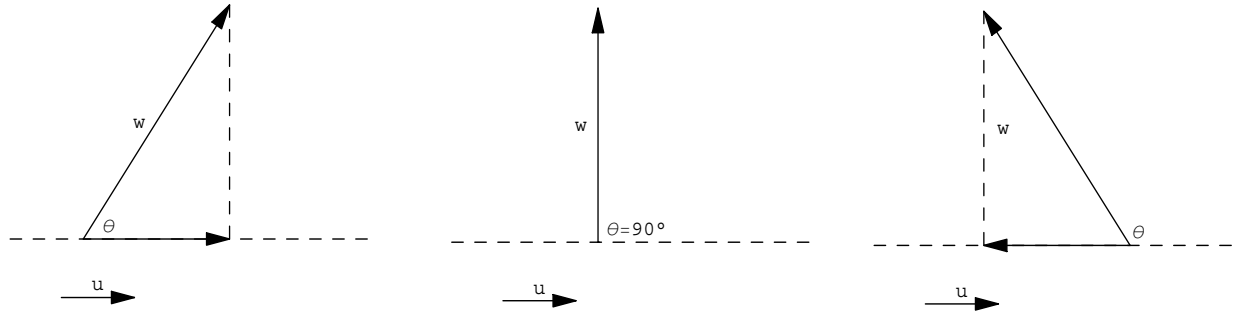
Suppose that \mathbf{v} and \mathbf{w} are vectors, and that we want to measure the extent to which \mathbf{w} points in the direction of \mathbf{v} . There are two ways to do this. One results in a vector, the other in a scalar.

Visualizing Projection. Remember that vectors are “free,” that is, they can be parallel translated. We often picture two vectors with their tails together, or with the tail of one at the point of the other, but in a complicated diagram this is sometimes inconvenient. Here is how to visualize projection without moving the vectors. Suppose that \mathbf{w} and \mathbf{v} are as pictured in the following three cases.



In each case, suppose ℓ is a line parallel to \mathbf{v} (this requires \mathbf{v} to be non-zero). Drop perpendiculars from the tail and point of \mathbf{w} to ℓ at points P and Q , respectively. The vector projection of \mathbf{w} in the direction of \mathbf{v} , denoted by $\text{Proj}_{\mathbf{v}}\mathbf{w}$, is the vector \overrightarrow{PQ} . Note that this is a multiple of \mathbf{v} . The scalar projection of \mathbf{w} in the direction of \mathbf{v} is the signed length \overrightarrow{PQ} : it is $\|\overrightarrow{PQ}\|$ when \overrightarrow{PQ} is a positive multiple of \mathbf{v} , and it is $-\|\overrightarrow{PQ}\|$ when \overrightarrow{PQ} is a negative multiple of \mathbf{v} . In the middle picture P and Q are the same point, the vector projection is the vector $\mathbf{0}$, and the scalar projection is the scalar 0.

Formulas. Let $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$ so that \mathbf{u} is the unit vector pointing in the direction of \mathbf{v} . Consider the three situations in which the angle between \mathbf{w} and \mathbf{u} is less than 90° , equal to 90° , and greater than 90° .



In the first and third cases, drop a perpendicular to form a right triangle as shown. The side of the triangle parallel to \mathbf{u} has length $\|\mathbf{w}\| |\cos \theta|$, where θ is the angle between \mathbf{w} and \mathbf{u} . Note that $\cos \theta > 0$ in the first case and $\cos \theta < 0$ in the third. In the second case $\cos \theta = 0$. In all three cases we see that the quantity $\|\mathbf{w}\| \cos \theta$ is the scalar projection, since it has the correct magnitude and sign. Furthermore, $(\|\mathbf{w}\| \cos \theta)\mathbf{u}$ is the vector projection.

Since \mathbf{u} is a unit vector we have

$$\mathbf{w} \cdot \mathbf{u} = \|\mathbf{w}\| \|\mathbf{u}\| \cos \theta = \|\mathbf{w}\| \cos \theta,$$

and so the scalar projection is exactly the value of $\mathbf{w} \cdot \mathbf{u}$. Thus the scalar and vector projections are, respectively, $\mathbf{w} \cdot \mathbf{u}$ and $(\mathbf{w} \cdot \mathbf{u})\mathbf{u}$. Substituting $\mathbf{v}/\|\mathbf{v}\|$ in for \mathbf{u} , we have

$$\text{scalar projection: } \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|}$$

and

$$\text{Proj}_{\mathbf{v}} \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{u} = \left(\mathbf{w} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \right) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

Uses of Projection. Scalar and vector projections are used in similar situations. Which one you use depends on the type of result you need.

Example 1. Given the line $2x + 3y = 1$ and point $P(-2, -1)$, find the signed distance from P to the line, where the positive side of the line is above and to the right. (See the picture. Note that you don't need an accurate picture for this specific line and point, a generic picture suffices and is often better.)

A normal vector to the line pointing in the positive direction is $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j}$. A point on the line is $Q(-1, 1)$. The signed distance is the scalar projection of \overrightarrow{QP} in the direction \mathbf{n} , given by

$$\frac{\overrightarrow{QP} \cdot \mathbf{n}}{\|\mathbf{n}\|} = \frac{(-1\mathbf{i} - 2\mathbf{j}) \cdot \mathbf{n}}{\|\mathbf{n}\|} = \frac{-8}{\sqrt{13}} = -\frac{8\sqrt{13}}{13}.$$

The distance from P to the line is $8\sqrt{13}/13$, and it is on the side opposite the direction pointed by \mathbf{n} .

Example 2. Given the same line and point, find the point R on the line closest to P . This question requires more, and so we expect to have to work a bit harder.

From the picture we see that

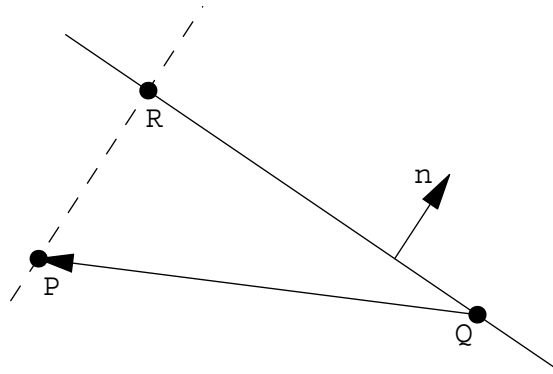
$$\overrightarrow{PR} = \text{Proj}_{\mathbf{n}} \overrightarrow{PQ}$$

(note that I'm using \overrightarrow{PQ} instead of \overrightarrow{QP}). We have

$$\overrightarrow{PR} = \text{Proj}_{\mathbf{n}} \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} = \frac{8}{13} \mathbf{n} = \frac{16}{13} \mathbf{i} + \frac{24}{13} \mathbf{j}.$$

It follows that

$$R = P + \overrightarrow{PR} = (-2, -1) + \left(\frac{16}{13} \mathbf{i} + \frac{24}{13} \mathbf{j} \right) = \left(-\frac{10}{13}, \frac{11}{13} \right).$$



Example 3. Another way to determine a line is by a point on the line and a vector parallel to the line. (Note that in three or more dimensions this is how lines are typically given.) For example, the line in the previous examples could have been determined by the point Q and the vector $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}$, which is parallel to the line (not pictured). In this case the point R can be located projecting parallel to the line as

$$R = Q + \overrightarrow{QR} = Q + \text{Proj}_{\mathbf{v}} \overrightarrow{QP} = (-1, 1) + \left(\frac{3}{13} \mathbf{i} - \frac{2}{13} \mathbf{j} \right) = \left(-\frac{10}{13}, \frac{11}{13} \right).$$

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