SUMMARY OF PATH INTEGRALS AND RELATED THEOREMS

Math 225 Chapter 6 of Colley's Text

Chapter 6 of Colley's text introduces path integrals and explores their consequences. This pulls together a lot of things we have done so far, and introduces several new ideas as well. It can be a bit overwhelming, and so here is a summary. Several coordinate formulas are stated in $\mathbb{R}^2$, but generalize to $\mathbb{R}^3$ in an obvious way. A few things only work in $\mathbb{R}^2$. They either have no generalization to $\mathbb{R}^3$, or the generalization to $\mathbb{R}^3$ is not obvious.

Path Integrals with Respect to Arc-Length (§6.1)

If C is a path in E^n and f is a function on C, we can define $\int_C f\,ds$.

Examples and Applications.

(1) $L = \int_C ds$ is the length of C. (§3.2)

(2) If δ is mass density (mass per unit length) along C, then $M = \int_C \delta\,ds$ is the mass of C.

(3) In $\mathbb{R}^2$ the centroid of C is the point $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{L}\int_C x\,ds$ and $\bar{y} = \frac{1}{L}\int_C y\,ds$. These are the average values of x and y on C. The integrals in these expressions are the moments of C.

(4) In $\mathbb{R}^2$ the center of mass of C is the point $(x_{\text{cm}}, y_{\text{cm}})$ where $x_{\text{cm}} = \frac{1}{M}\int_C x\delta\,ds$ and $y_{\text{cm}} = \frac{1}{M}\int_C y\delta\,ds$. These are weighted average values of x and y on C, weighted by δ. The integrals in these expressions are the moments of the mass distribution.

Path Integrals of a Vector Field (§6.1)

If C is a path in E^n and $\mathbf{F}$ is a vector field along C, we can define $\int_C \mathbf{F}\cdot dX$. Here X represents a point in E^n. Then dX represents an infinitesimal displacement along C, and so is a vector. Since ds is the length of dX, we have $dX/ds = \mathbf{T}$, where $\mathbf{T}$ is the unit tangent vector pointing in the direction of motion along C (from §3.2). We get

$$\int_C \mathbf{F}\cdot dX = \int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

In $\mathbb{R}^2$ we have $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, where P and Q are the coordinate functions of $\mathbf{F}$, $X = (x, y)$, and $dX = dx\,\mathbf{i} + dy\,\mathbf{j}$, and so

$$\int_C \mathbf{F}\cdot dX = \int_C P\,dx + Q\,dy.$$

In $\mathbb{R}^2$, let $\mathbf{n} = -\mathbf{T}_\perp$, so that $\mathbf{n}$ is a unit vector pointing to the right as you move along the curve (this is the opposite of $\mathbf{N}$ used in Section 3.2). Then we can define $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$. (This does not generalize directly to $\mathbb{R}^3$ because there is no perp operator in $\mathbb{R}^3$.)

Applications.

(1) If $\mathbf{F}$ is a force acting on a particle that moves along C, then $\int_C \mathbf{F}\cdot dX = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is the work done by $\mathbf{F}$.

(2) If $\mathbf{F}$ is the velocity of a fluid flow, then $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is a measure of how the flow agrees with the direction of the curve. In particular, if C is the boundary of some region R, then $\oint_C \mathbf{F}\cdot\mathbf{T}\,ds$ is a measure of the instantaneous circulation of the flow around the boundary of the region. Green's Theorem (§6.2) can then be used to give an interpretation of $\operatorname{curl}\mathbf{F}$.

(3) In $\mathbb{R}^2$, if $\mathbf{F}$ is the velocity of a fluid flow, then $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$ is a measure of how the flow crosses the curve in the direction of $\mathbf{n}$. This is called the *flux* of $\mathbf{F}$ across the curve (§6.2). In particular, if C is the boundary of some region R oriented so that $\mathbf{n}$ is the outward-pointing unit normal, then $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ is a measure of the rate at which the fluid is escaping the region. Green's Theorem (§6.2) can then be used to give an interpretation of $\operatorname{div}\mathbf{F}$. In $\mathbb{R}^3$ there is a notion of flux across a surface, but not across a curve (§7.2).

Dependence on Orientation (§6.1)

The integral $\int_C f\,ds$ does not depend on orientation; the integral $\int_C \mathbf{F}\cdot dX$ does. More specifically, if $\tilde{C}$ represents C with the opposite orientation, then

$$\int_{\tilde{C}} f\,ds = \int_C f\,ds \qquad \text{and} \qquad \int_{\tilde{C}} \mathbf{F}\cdot dX = -\int_C \mathbf{F}\cdot dX.$$

This is part of Theorem 1.5.

Evaluation of Path Integrals (§6.1)

Suppose C is the image of $\gamma\colon [a,b] \to E^n$, assumed to be piecewise C^1. One can formally take $X = \gamma(t)$ and $dX = \gamma'(t)\,dt$ to get

$$\int_C \mathbf{F}\cdot dX = \int_C \mathbf{F}(X)\cdot dX = \int_a^b \mathbf{F}\big(\gamma(t)\big)\cdot\gamma'(t)\,dt.$$

In coordinates we have $x = x(t)$, $y = y(t)$, $dx = x'(t)\,dt$, and $dy = y'(t)\,dt$, and so

$$\int_C P\,dx + Q\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \Big(P\big(x(t),y(t)\big)x'(t) + Q\big(x(t),y(t)\big)y'(t)\Big)\,dt.$$

Note that in some cases the parameter can be taken to be x or y. For example, if the curve is given by $y = y(x)$ for $a \le x \le b$, then $dy = y'(x)\,dx$ and

$$\int_C P\,dx + Q\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \Big(P\big(x,y(x)\big) + Q\big(x,y(x)\big)y'(x)\Big)\,dx.$$

To evaluate a flux integral (§6.2) in $\mathbb{R}^2$ (but not in $\mathbb{R}^3$), use the fact that $\mathbf{F}\cdot\mathbf{n} = \mathbf{F}_\perp \cdot \mathbf{n}_\perp = \mathbf{F}_\perp \cdot \mathbf{T}$:

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C \mathbf{F}_\perp\cdot\mathbf{T}\,ds = \int_C \mathbf{F}_\perp\cdot dX = \int_C -Q\,dx + P\,dy = \int_a^b \mathbf{F}\big(\gamma(t)\big)_\perp\cdot\gamma'(t)\,dt.$$

For path integrals with respect to arc length we have $ds = \frac{ds}{dt}\,dt = \|\gamma'(t)\|\,dt$ or $ds = \|dX\| = \|\gamma'(t)\,dt\| = \|\gamma'(t)\|\,dt$, and so

$$\int_C f\,ds = \int_C f(X)\,ds = \int_a^b f\big(\gamma(t)\big)\,\|\gamma'(t)\|\,dt.$$

In coordinates we have $ds = \sqrt{dx^2 + dy^2} = \sqrt{x'(t)^2 + y'(t)^2}\,dt$, and so

$$\int_C f\,ds = \int_C f(x,y)\,ds = \int_a^b f\big(x(t),y(t)\big)\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

If the curve is given by $y = y(x)$ for $a \le x \le b$, then

$$\int_C f\,ds = \int_C f(x,y)\,ds = \int_a^b f\big(x,y(x)\big)\sqrt{1 + y'(x)^2}\,dx.$$

When $f = 1$, the length of the curve is then $\int_a^b \sqrt{1 + y'(x)^2}\,dx$, a formula you may remember from Math 112.

Path integrals and Green's Theorem (§6.2)

Green's Theorem and Variations. *Let R be a bounded region in $\mathbb{R}^2$ with piecewise C^1 boundary ∂R, oriented in the positive direction. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a C^1 vector field on R. Then*

$$\oint_{\partial R} P\,dx + Q\,dy = \oint_{\partial R} \mathbf{F}\cdot dX = \oint_{\partial R} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R \operatorname{curl}\mathbf{F}\,dA$$

and

$$\oint_{\partial R} \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \operatorname{div}\mathbf{F}\,dA.$$

The last variation is known as the Divergence Theorem.

Green's Theorem is an important generalzation of the Fundamental Theorem of Calculus, $f(b) - f(a) = \int_a^b f'(x)\,dx$. Both say that an oriented sum of a quantity on the boundary is equal to the sum of some derivative of the quantity on the interior.

Recall the interpretations of $\oint_{\partial R} \mathbf{F} \cdot \mathbf{T}\,ds$ and $\oint_{\partial R} \mathbf{F} \cdot \mathbf{n}\,ds$ in the second section above. Green's Theorem then gives interpretations to $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$ as measures of rotation and expansion.

Green's Theorem and its variations generalize to $\mathbb{R}^3$ in two ways, known as the theorems of Gauss and Stokes (§7.3). They generalize further to fairly general multi-dimensional surfaces and solids, and are known collectively simply as Stokes' Theorem (§8.3). It is an important theorem of modern mathematics.

Generalized Stokes' Theorem (or the FTC on steroids). *Let M be an oriented n-dimensional manifold with boundary ∂M. Let α be a differential form of degree $n-1$ on M. Then*

$$\int_{\partial M} \alpha = \int_M d\alpha.$$

This is the Fundamental Theorem of Calculus (Math 111 version and for path integrals, Theorem 1 below), Green's Theorem, the Divergence Theorem, the theorems of Gauss and Stokes, and more, all rolled into one!

Path Integral Version of the Fundamental Theorem of Calculus (§6.3)

Theorem 1. *Suppose that U is an open set in E^n, that $f\colon U \to \mathbb{R}$ is C^1, and that C is a piecewise C^1 oriented path in U from A to B. Then*

$$\int_C \nabla f \cdot dX = f(X)\Big|_A^B = f(B) - f(A).$$

This is part of Theorem 3.3 in Colley.

Path Integrals and Conservative Vector Fields (§6.3)

Looking through my collection of mathematics and physics books, I find that mathematicians and physicists give different definitions for a conservative vector field!

Mathematician's Definition. Suppose that U is an open set in E^n and that $\mathbf{F}$ is a continuous vector field on U. We say that $\mathbf{F}$ is *conservative* if $\mathbf{F}$ is the gradient of some function on U, that is, if there is a function $f\colon U \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$.

Note: If $\mathbf{F} = \nabla f$, then f is called a *potential* for $\mathbf{F}$ (physicists call $-f$ a potential).

Physicist's Definition. Suppose that U is an open set in E^n and that $\mathbf{F}$ is a continuous vector field on U. They say that $\mathbf{F}$ is *conservative* if for every piecewise C^1 path C in U, the integral $\int_C \mathbf{F} \cdot dX$ depends only on the endpoints of C. Another way to say this is that for every pair of piecewise C^1 paths C and $\tilde{C}$ in U joining the same two points we have $\int_C \mathbf{F}\cdot dX = \int_{\tilde{C}} \mathbf{F}\cdot dX$. (Note: Sometimes this condition on $\int_C \mathbf{F}\cdot dX$ is described by saying that the integral is independent of path. This is really a misstatement. The integral does depend on the path somewhat; it depends on its endpoints.)

These two definitions are really just different points of view because of the following theorem.

Theorem 2. *Suppose that U is an open set in E^n and that $\mathbf{F}$ is a continuous vector field on U. Then the following statements about $\mathbf{F}$ are equivalent.*

(1) *$\mathbf{F}$ is the gradient of some function. (Math definition of conservative)*

(2) *For every piecewise C^1 path C in U, the integral $\int_C \mathbf{F}\cdot dX$ depends only on the endpoints of C. (Physics definition of conservative)*

(3) *For every piecewise C^1 loop C in U, we have $\int_C \mathbf{F}\cdot dX = 0$.*

Note that (1) $\iff$ (2) is part of Theorem 3.3 in Colley and (2) $\iff$ (3) is Theorem 3.2. The next theorem says that it is *almost* the case that a vector field $\mathbf{F}$ is conservative if and only if $\operatorname{curl}\mathbf{F} = 0$.

Theorem (Test for Conservative Vector Fields). *Suppose that U is an open set in E^2 or E^3 and that $\mathbf{F}$ is a C^1 vector field on U.*

(1) *If $\mathbf{F}$ is conservative, then* $\operatorname{curl}\mathbf{F} = 0$.

(2) *Suppose U is simply connected. Then $\mathbf{F}$ is conservative if and only if* $\operatorname{curl}\mathbf{F} = 0$.

Remarks. In $\mathbb{R}^2$ we have $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, where P and Q are the coordinate functions of $\mathbf{F}$. Then $\operatorname{curl}\mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

- Part (1) of the theorem is just a restatement of something we saw earlier, namely, if $\mathbf{F}$ is a gradient, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. (This generalizes in $\mathbb{R}^3$ to three equations, since $\operatorname{curl}\mathbf{F}$ is a vector.)
- The expression $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ appears in the double integral in Green's Theorem (§6.2). (In $\mathbb{R}^3$ the coefficients of $\operatorname{curl}\mathbf{F}$ appear in a coordinate version of Stokes' Theorem (§7.3)), a formula that is truly scary and does not often appear in undergraduate texts!)

If the set U is not simply connected (in $\mathbb{R}^2$ this happens if it has one or more holes) and $\operatorname{curl}\mathbf{F} = 0$, the vector field $\mathbf{F}$ might be a gradient, but it need not be. For a discussion on this, see the handout "A Curl-Free Vector Field that is not a Gradient."

Note to physics majors. When I took physics, I remember when determining that $\operatorname{curl}\mathbf{F} = 0$ for a vector field, the next step was almost always to assume that $\mathbf{F}$ has a potential—it was hard to imagine it *not* having a potential. Learning that $\mathbf{F}$ does not have to have a potential was disconcerting. Hopefully this will cause you to ponder why the important physical vector fields that are curl-free (gravitational, electric) have potentials, what physics would be like if they didn't, what *physical* properties cause this, and how we know.

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