PATH INTEGRALS WITH RESPECT TO ARC-LENGTH

There are *two* types of path integrals, path integrals with respect to arc-length (scalar path integrals) and integrals of a vector field along a path (vector path integrals). Path integrals with respect to arc-length are used in physics to compute the moments and center of mass of a wire.

Let C be a path in $\mathbb{R}^3$, and let f be a function along the curve. We want to make sense out of

$$\int_C f\,ds.$$

The idea is that at each point along the curve, an infinitesimal length ds of the curve is multiplied by the value of the function f at that point, and the result is summed (integrated) along the curve.

Here are some common examples of integrals with respect to arc-length.

(1) $L = \int_C ds$ is the length of the curve, in which the function being integrated is the constant 1.

(2) If δ is the mass density per unit length, then $M = \int_C \delta\,ds$ is the mass of the curve (think of it as a wire with variable mass density).

(3) $\int_C x\,ds$ is the geometric x-moment of the curve (or the moment about the yz-plane). It is, effectively, the sum of the x-coordinates of the points on the curve. If L is the length of the curve, then $x_c = \frac{1}{L}\int_C x\,ds$ is the average x-coordinate. If you do the same thing with y and z, then (x_c, y_c, z_c) is called the *centroid* of the curve, which should be thought of as the average position of the points on the curve.

(4) If δ is the mass density, then $\int_C x\delta\,ds$ is the physical x-moment of the wire. If M is the mass of the wire, then $x_{cm} = \frac{1}{M}\int_C x\delta\,ds$ is x-coordinate of the average location of mass (the center of mass). Doing the same thing with y and z you get the center of mass (x_{cm}, y_{cm}, z_{cm}).

The function f is a function of position on the curve. Quite often it can be written as a function of position in $\mathbb{R}^3$, that is $f\colon \mathbb{R}^3 \to \mathbb{R}$. (This is the case in $\int_C x\,ds$, where $f(x,y,z) = x$.) Suppose that $(x,y,z) = (x(t), y(t), z(t)) = \gamma(t)$, $a \le t \le b$ is a parameterization of C. Working intuitively, as when computing arc-length, we have

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt = \|\gamma'(t)\|\;dt.$$

Then

$$\int_C f\,ds = \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2 + \left(\tfrac{dz}{dt}\right)^2}\,dt = \int_a^b f(\gamma(t))\,\|\gamma'(t)\|\;dt.$$

This "substitution" turns $\int_C f\,ds$ into a Math 111 integral in which the variable of integration is the parameter. With obvious modifications, this can be done in any $\mathbb{R}^n$.

Example. Let C be the curve parameterized by $\gamma(t) = (x(t), y(t)) = (t - \sin t, 1 - \cos t)$, for $0 \leq t \leq 2\pi$. This is one arch of a cycloid. We will find its centroid. This example is worked out in the *Mathematica* notebook PathIntegral.nb in the Math 225 folder. Take some time to go through the computations both by hand seeing how it is done using *Mathematica*.

The term ds occurs in all of the integrals, so compute it first:

$$ds = \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\,dt = \|\gamma'(t)\|\,dt = \sqrt{(1-\cos t)^2 + \sin^2 t}\,dt = \sqrt{2 - 2\cos t}\,dt.$$

The length of C is then

$$L = \int_C ds = \int_0^{2\pi} \|\gamma'(t)\|\,dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt = 8.$$

To compute the geometric x-moment of C (the moment about the y-axis), put in $x = x(t) = t - \sin t$, getting

$$\int_C x\,ds = \int_0^{2\pi} x(t)\,\|\gamma'(t)\|\,dt = \int_0^{2\pi} (t - \sin t)\sqrt{2 - 2\cos t}\,dt = 8\pi.$$

The x-coordinate of the centroid ("average" value of x on the curve) is then

$$x_c = \frac{1}{L}\int_C x\,ds = \pi,$$

which is not surprising because of the symmetry of the curve. In fact, if this computation could have been skipped by observing the symmetry! The y-coordinate is more interesting. The moment about the x-axis is

$$\int_C y\,ds = \int_0^{2\pi} y(t)\,\|\gamma'(t)\|\,dt = \int_0^{2\pi} (1 - \cos t)\sqrt{2 - 2\cos t}\,dt = 32/3,$$

and so

$$y_c = \frac{1}{L}\int_C y\,ds = \frac{4}{3}.$$

The centroid is the point $(x_c, y_c) = (\pi, 4/3)$.

As with arc-length integrals, integrals with respect to arc-length often have integrands that don't have elementary antiderivatives. In this case, the only way to evaluate them is to approximate them numerically. You should always get exact answers if possible, and resort to numerical approximations only when necessary.

How the Two Types of Path Integrals are Different

It is important to note that a path integral with respect to arc-length does not depend on the orientation, or direction, of the path. If $-C$ denotes the path C with its orientation reversed, then

$$\int_C f\,ds = \int_{-C} f\,ds.$$

On the other hand, if $\mathbf{F} = \mathbf{F}(x, y, z)$ is a continuous vectorfield on $\mathbb{R}^3$, the path integral $\int_C \mathbf{F} \cdot d\mathbf{x}$, *does* depend on orientation, since

$$\int_C \mathbf{F}\cdot d\mathbf{x} = -\int_{-C} \mathbf{F}\cdot d\mathbf{x},$$

as is discussed in our text.

How the Two Types of Path Integrals are Related

Let's suppose that C is parameterized by arc-length, that is, C is the image of a parameterized curve $\gamma\colon [a, b] \to \mathbb{R}^3$ that is a function of s, where s measures distance along the path from some point. In this case, $\gamma'(s) = d\gamma/ds = \mathbf{T}$, where $\mathbf{T}$ is the unit tangent vector along the curve. We then have

$$\int_C \mathbf{F}\cdot d\mathbf{x} = \int_a^b \mathbf{F}(\gamma(s))\cdot\gamma'(s)\,ds = \int_a^b \mathbf{F}(\gamma(s))\cdot\mathbf{T}(\gamma(s))\,ds = \int_C \mathbf{F}\cdot\mathbf{T}\,ds,$$

that is, $\int_C \mathbf{F}\cdot d\mathbf{x} = \int_C f\,ds$, where f is the function $\mathbf{F} \cdot \mathbf{T}$ along the curve.

It's important to realize that even though we have written the integral of the vector field as an arc-length integral, it still depends on the orientation of the curve. If the orientation of the curve is reversed, then the unit vector $\mathbf{T}$ switches directions. Thus the function $f = \mathbf{F} \cdot \mathbf{T}$ in one direction is the negative of the function $\tilde{f} = \mathbf{F} \cdot \mathbf{T}$ in the opposite direction.

Additional Problems on Path Integrals With Respect to Arc Length
Math 225

You will need *Mathematica* on most of these.

1. Evaluate the following path integrals, where C is as indicated:
   (a) $\int_C (x^3 + y)\,ds$; C: $x = 3t$, $y = t^3$, $0 \le t \le 1$
   (b) $\int_C xy^{2/5}\,ds$; C: $x = t/2$, $y = t^{5/2}$, $0 \le t \le 1$
   (c) $\int_C xyz\,ds$; C is the line segment from $(0,0,0)$ to $(1,2,3)$
   (d) $\int_C (xy + z)\,ds$; C: $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 2\pi$

2. For each of the following thin wires with given mass density, a) find the length and centroid, b) find the mass and center of mass, c) plot the curve, its centroid, and its center of mass.
   (a) The wire lies along the parabola $y = 4 - x^2$ in $\mathbf{R^2}$ between $(-2, 0)$ and $(2, 0)$. The density δ is proportional to the distance from the y-axis.
   (b) The wire lies along the graph of $y = \sqrt{1 - x^2}$ in $\mathbf{R^2}$ between $(-1, 0)$ and $(1, 0)$. The density δ is proportional to the distance from the x-axis.
   (c) The wire lies along the helix $x = \cos t$, $y = \sin t$, $z = t$, $0 \le t \le 3\pi$, in $\mathbf{R^3}$. The density δ is proportional to the distance from the (x, y)-plane.