TWO VERY USEFUL THINGS FROM LINEAR ALGEBRA

Math 225

Find these two things in your linear algebra book!

Finding Coefficients Relative to an Orthogonal Basis

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis for a vector space. Given a vector $\mathbf{w}$ in the space, one often needs to write $\mathbf{w}$ as a linear combination of the basis vectors, that is, one needs to find scalars a_1, $\ldots$, a_n such that

$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n.$$

(The fact that these scalars exist and are unique is a consequence of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ forming a basis.) In general, finding these coefficients can be very messy—it involves a linear system of n equations in n unknowns. However, if the basis vectors are mutually orthogonal, there is a much easier approach. An example will illustrate the general process.

Consider the vectors

$$\mathbf{v}_1 = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \qquad \mathbf{v}_2 = \mathbf{i} - 2\mathbf{j}, \qquad \text{and} \qquad \mathbf{v}_3 = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$$

in $\mathbb{R}^3$. These are mutually orthogonal (which you should verify). Since there are three of them and $\mathbb{R}^3$ is a three-dimensional vector space, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis. Let $\mathbf{w} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$. We want to find scalars a_1, a_2, a_3 such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3.$$

To find the coefficients, dot both sides of this with $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, successively. Dotting both sides with $\mathbf{v}_1$ yields

$$\mathbf{w} \cdot \mathbf{v}_1 = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3) \cdot \mathbf{v}_1 = a_1(\mathbf{v}_1 \cdot \mathbf{v}_1).$$

Note that two of the three terms are zero because $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$. Plugging in the specific vectors yields

$$19 = a_1(6), \qquad \text{and so} \qquad a_1 = \frac{19}{6}.$$

Similarly,

$$2 = \mathbf{w} \cdot \mathbf{v}_2 = a_2(\mathbf{v}_2 \cdot \mathbf{v}_2) = a_2(5) \qquad \text{and} \qquad -11 = \mathbf{w} \cdot \mathbf{v}_3 = a_3(\mathbf{v}_3 \cdot \mathbf{v}_3) = a_3(30),$$

and so $a_2 = 2/5$ and $a_3 = -11/30$. Thus,

$$\mathbf{w} = \frac{19}{6}\mathbf{v}_1 + \frac{2}{5}\mathbf{v}_2 - \frac{11}{30}\mathbf{v}_3.$$

This is *much* easier than solving the system of equations, but it *only* works for orthogonal bases. (Think about what would happen above if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ were not orthogonal.) This is why your linear algebra book spent time on orthonormal bases. (An orthonormal basis is one in which the basis vectors are mutually orthogonal and unit vectors. The most important feature of an orthonormal basis is that the vectors are orthogonal, not that they are unit vectors.)

If you go back through the derivation above without using the specific vectors, you will get

$$\mathbf{w} = \frac{\mathbf{w} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\mathbf{w} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{\mathbf{w} \cdot \mathbf{v}_3}{\|\mathbf{v}_3\|^2}\mathbf{v}_3.$$

This is simply the sum of the projections of $\mathbf{w}$ in the directions of the basis vectors. Again, this works because the vectors are orthogonal. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are unit vectors in addition to being orthogonal, then the formula simplifies to

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w} \cdot \mathbf{v}_2)\mathbf{v}_2 + (\mathbf{w} \cdot \mathbf{v}_3)\mathbf{v}_3.$$

The Matrix of a Linear Transformation

If V and W are vector spaces, recall that a function $L : V \to W$ is *linear* if

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}) \qquad \text{and} \qquad L(a\mathbf{x}) = aL(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y}$ in V and all a in $\mathbb{R}$.

Suppose $L\colon \mathbb{R}^k \to \mathbb{R}^\ell$ is linear. Then L is represented by matrix multiplication, that is, there is an $\ell \times k$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^k$. Here is the procedure for computing the matrix A.

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be the standard basis of $\mathbb{R}^k$.
- Compute $L(\mathbf{e}_1), \ldots, L(\mathbf{e}_k)$. These are vectors in $\mathbb{R}^\ell$.
- Write the vectors from the previous step as columns.
- Use the columns to construct the matrix.

In short, $L(\mathbf{e}_j)$ is the j^{th} column of A.

Example. Define $L\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $L(x, y, z) = (3x - 2y + 5z, -2x + y + 4z)$. This is linear (which you might try verifying).

- The standard basis of $\mathbb{R}^3$ consists of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.
- $L(\mathbf{i}) = L(1, 0, 0) = (3, -2)$, $L(\mathbf{j}) = L(0, 1, 0) = (-2, 1)$, and $L(\mathbf{k}) = L(0, 0, 1) = (5, 4)$.
- The columns of A are $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.
- $A = \begin{pmatrix} 3 & -2 & 5 \\ -2 & 1 & 4 \end{pmatrix}$.

You can see the coefficients of the matrix in the formula for L, but you won't always have a formula!

Sometimes it is useful to use bases other than the standard bases. Sometimes you have to if you are using vector spaces that don't have standard bases. In this case the process is similar. Suppose V and W are vector spaces and that $L\colon V \to W$ is linear.

- Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ form a basis for V and that $\mathbf{w}_1, \dots, \mathbf{w}_\ell$ form a basis for W. To find the matrix A that represents L relative to these bases:
- Compute $L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)$. These are vectors in W.
- Write each vector in the previous part in terms of the basis for W, that is, express each as a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_\ell$. (This is easy if $\mathbf{w}_1, \dots, \mathbf{w}_\ell$ are orthogonal—see the first part of the handout.)
- Write the *coefficients* of each $L(\mathbf{v}_j)$ found in the previous part as a column.
- Use the columns to construct the matrix.

In short, $L(\mathbf{v}_j)$ determines the j^{th} column of A. To get the column, express $L(\mathbf{v}_j)$ in terms of the basis for W. The column consists of the coefficients.

Example. Let L be the same linear map as in the previous example. For $\mathbb{R}^3$ use the basis

$$\mathbf{v}_1 = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \qquad \mathbf{v}_2 = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k}, \qquad \mathbf{v}_3 = \mathbf{j} - \mathbf{k}.$$

For $\mathbb{R}^2$ use the basis

$$\mathbf{w}_1 = 2\mathbf{i} + 3\mathbf{j}, \qquad \mathbf{w}_2 = -3\mathbf{i} + 2\mathbf{j}.$$

Note this basis is orthogonal.

- $$\begin{gathered} L(\mathbf{v}_1) = L(-1,2,1) = (-2,8) = -2\mathbf{i} + 8\mathbf{j} \\ L(\mathbf{v}_2) = L(3,-1,-2) = (1,-15) = \mathbf{i} - 15\mathbf{j} \\ L(\mathbf{v}_3) = L(0,1,-1) = (-7,-3) = -7\mathbf{i} - 3\mathbf{j} \end{gathered}$$
- Using the method of the first part of the handout, show that

$$L(\mathbf{v}_1) = -2\mathbf{i} + 8\mathbf{j} = \frac{20}{13}\mathbf{w}_1 + \frac{22}{13}\mathbf{w}_2$$

$$L(\mathbf{v}_2) = \mathbf{i} - 15\mathbf{j} = -\frac{43}{13}\mathbf{w}_1 - \frac{33}{13}\mathbf{w}_2$$

$$L(\mathbf{v}_3) = -7\mathbf{i} - 3\mathbf{j} = -\frac{23}{13}\mathbf{w}_1 + \frac{15}{13}\mathbf{w}_2$$

- The columns are $\begin{pmatrix} 20/13 \\ 22/13 \end{pmatrix}$, $\begin{pmatrix} -43/13 \\ -33/13 \end{pmatrix}$, and $\begin{pmatrix} -23/13 \\ 15/13 \end{pmatrix}$.
- The matrix is
$$\begin{pmatrix} 20/13 & -43/13 & -23/13 \\ 22/13 & -33/13 & 15/13 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 20 & -43 & -23 \\ 22 & -33 & 15 \end{pmatrix}.$$

  Note: It's the *same* linear map, but a *different* matrix, because we are using different bases.

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