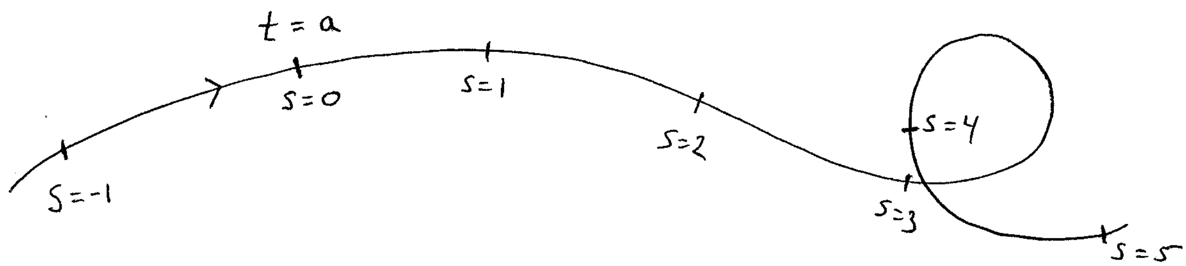
GEOMETRY OF CURVES IN E^2

This is a summary of the formulas on the geometry of curves in E^2. Note that my approach to this is slightly different than the one in the book. The difference is in the way the unit normal and curvature are defined.

Let $\gamma\colon \mathbb{R} \to E^2$ be a curve that is twice continuously differentiable, that is, the velocity and acceleration vectors, $\mathbf{v} = \gamma'$ and $\mathbf{a} = \gamma''$ are defined and continuous as vector-valued functions of t. (Note: The domain of γ doesn't have to be $\mathbb{R}$, but can be any interval in $\mathbb{R}$.) Let $X = \gamma(t)$ be a typical point on the curve.

Arc-Length. Arc-length measures distance along the curve. We can start measuring at any point, that is, at any time. If we start measuring at $t = a$, then we have $s = 0$ when $t = a$. Note that s increases in the direction of motion. (The direction of motion is called the *orientation* of the curve.)

$$s(t) = \int_a^t \|\mathbf{v}(\tau)\|\, d\tau \qquad\qquad \frac{ds}{dt} = \|\mathbf{v}\|$$



Arc-length can be used as a parameter. This means that quantities that depend on position on the curve, namely X, $\mathbf{v}$, $\mathbf{a}$, and κ, can be thought of as functions of s as well as t. A quantity that is differentiable as a function of t is also differentiable with respect to s at points where $\mathbf{v} \neq \mathbf{0}$. The chain rule tells us how to switch between the two types of derivatives:

$$\frac{du}{ds} = \frac{du/dt}{ds/dt} = \frac{du/dt}{\|\mathbf{v}\|}, \qquad \frac{d\mathbf{w}}{ds} = \frac{d\mathbf{w}/dt}{ds/dt} = \frac{d\mathbf{w}/dt}{\|\mathbf{v}\|}, \qquad \frac{dX}{ds} = \frac{dX/dt}{ds/dt} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \mathbf{T}.$$

The assumption that $\mathbf{v} \neq \mathbf{0}$ will be made for the rest of these notes.

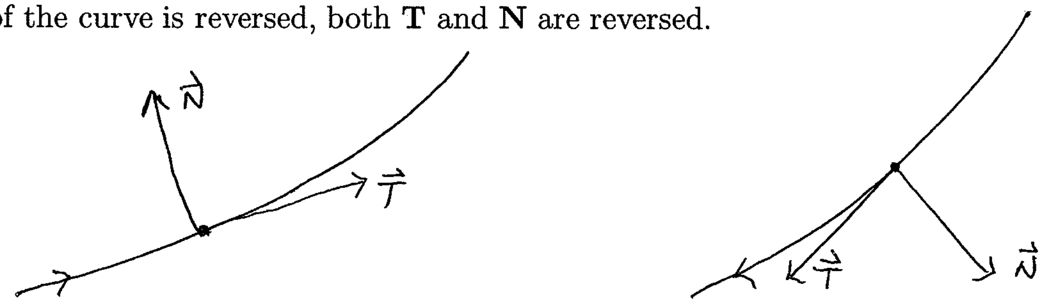
Unit Tangent and Normal Vectors. The unit tangent vector is defined by

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{dX}{ds}.$$

So far, everything up to this point can be done for curves in E^n. However, the definition of the unit normal vector depends on being in two dimensions (and so then does everything that depends on the unit normal):

$$\mathbf{N} = \mathbf{T}_\perp = \frac{\mathbf{v}_\perp}{\|\mathbf{v}\|} \qquad \text{(different from the book).}$$

$\mathbf{T}$ is tangent to the curve and points in the direction of the orientation. $\mathbf{N}$ is perpendicular to the curve, pointing to the left as you move along the curve. Note that if the orientation of the curve is reversed, both $\mathbf{T}$ and $\mathbf{N}$ are reversed.



Curvature. Since $\mathbf{T}$ has constant length, its derivatives with respect to both t and s are perpendicular to $\mathbf{T}$. Thus $d\mathbf{T}/ds$ is some multiple of $\mathbf{N}$. That multiple is the curvature, that is, curvature, κ, is defined by the equation

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}.$$

The curvature measures how fast the curve is changing direction. The larger κ is (and the longer $d\mathbf{T}/ds$ is), the sharper the curve is turning. A circle of radius r, oriented counterclockwise, has curvature $\kappa = 1/r$. A line has curvature $\kappa = 0$. (You should be able to prove these last two statements.

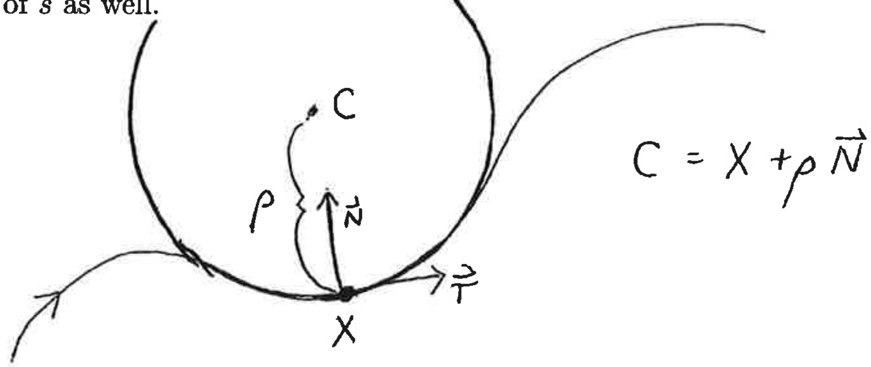
The vector $d\mathbf{T}/ds = \kappa\mathbf{N}$ is perpendicular to the curve and points towards the inside of the curve. Thus, if the curve is turning counterclockwise, this vector points in the same direction as $\mathbf{N}$, and κ is positive. If the curve is turning clockwise, this vector points in the opposite direction as $\mathbf{N}$, and κ is negative. (Using the definitions in the book, the curvature is always positive and $\mathbf{N}$ switches sides of the curve depending on the direction of turning.) This approach (as opposed to the one in the book) allows $\mathbf{N}$ to be computed more easily, and makes the curvature more meaningful, since it can distinguish between clockwise and counterclockwise.

Let φ be the angle determined by the equation $\mathbf{T} = \cos\varphi\,\mathbf{i} + \sin\varphi\,\mathbf{j}$. This is the angle between $\mathbf{T}$ and $\mathbf{i}$. (Note that then $\mathbf{T} = \mathbf{u}(\varphi)$, where $\mathbf{u}(\varphi)$ is the unit direction vector used in class.) It can be shown (a good chain rule exercise!) that $\kappa = d\varphi/ds$, thus the curvature measures the rate of change of the this angle. This says, perhaps even more convincingly, that curvature measures how fast the curve turns. Note that this formula also shows that

$\kappa > 0$ when the curve is turning counterclockwise and $\kappa < 0$ when the curve is turning clockwise.

Small curvature Large curvature Positive curvature Negative curvature

Radius of Curvature and Osculating Circle. The osculating circle is the circle that best fits the curve at a point on the curve. Its center lies on the line perpendicular to the curve at the point, and its radius is $|\rho|$, where $\rho = 1/\kappa$. The scalar ρ, which is a function of where you are on the curve, is called the radius of curvature even though it can be negative (when $\kappa < 0$). The center of the osculating circle is called the center of curvature. For a specific value of t the the center of curvature is the point $\gamma(t) + \rho(t)\mathbf{N}(t)$. As t varies, the center and radius of curvature change, and the center of curvature traces another curve, $X + \rho\mathbf{N}$, called the *evolute* of the original curve. Note that we can think of X, ρ, and $\mathbf{N}$ as being functions of s as well.



A Moving Frame. You want to visualize $\mathbf{T}$ and $\mathbf{N}$ moving along the curve (there is an animation of this in the *Mathematica* notebook Curvature.nb in the Math225 folder). As such, they are called a moving frame for the curve. We are used to using $\mathbf{i}$ and $\mathbf{j}$ as the standard basis for vectors in two dimensions. Since $\mathbf{T}$ and $\mathbf{N}$ are independent, they form a basis too. Thus we can express any vector in two dimensions as a linear combination of $\mathbf{T}$ and $\mathbf{N}$. Unlike $\mathbf{i}$ and $\mathbf{j}$, the vectors $\mathbf{T}$ and $\mathbf{N}$ depend on the geometry of the curve. Consequently, when a vector is expressed as a linear combination of $\mathbf{T}$ and $\mathbf{N}$, the coefficients will tell us something about how that vector is related to the geometry of the curve. The simplest examples are

$$\mathbf{v} = v\mathbf{T} \quad \text{and} \quad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \text{where} \quad v = \|\mathbf{v}\| = \frac{ds}{dt}.$$

A more substantial example is

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N}.$$

This very cool formula, which resolves acceleration into components tangent and normal to the curve, tells us how the physics of a moving particle is related to the geometry of the curve. The physical quantities on the righthand side are v and dv/dt, the speed and how fast it changes (not to be confused with acceleration). The geometric quantity is the curvature κ. The formula tells us that the component of $\mathbf{a}$ tangent to the curve changes the speed of the particle, and the component perpendicular to the curve causes the particle to turn. They combine to form the acceleration: it depends on both the speed of the particle and the geometry of the path. You should be able to derive this formula, and that will help you understand it. (Start by differentiating $\mathbf{v} = v\mathbf{T}$.)

Computations. Note that it takes one derivative to define $\mathbf{T}$, $\mathbf{N}$, and ds/dt, and a second derivative to define κ. The other operations relating them are algebraic. It seems reasonable to expect that these quantities will be algebraic combinations of $\mathbf{v}$ and $\mathbf{a}$. This is indeed the case. The formulas are

$$\mathbf{T} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \mathbf{N} = \mathbf{T}_\perp, \qquad \frac{ds}{dt} = \|\mathbf{v}\|, \qquad \text{and} \qquad \kappa = \frac{\mathbf{a}\cdot\mathbf{v}_\perp}{\|\mathbf{v}\|^3}.$$

The first two of these are definitions. The third is a consequence of the integral formula for the arc length function $s(t)$ and the Fundamental Theorem of Calculus (remember it?). You should be able to derive the formula for curvature. (Start by dotting the formula for $\mathbf{a}$ by $\mathbf{v}_\perp$.)

Observation. The parameter doesn't have to be thought of as time. Then $\mathbf{v}$ and $\mathbf{a}$ aren't velocity and acceleration, they're just the first and second derivatives with respect to the parameter. The curvature formula in this case becomes

$$\kappa = \frac{\gamma''\cdot\gamma'_\perp}{\|\gamma'\|^3}.$$

Examples.

(1) Consider the parabola $y = x^2$. This can be parameterized by letting $x = t$ (or just forget about t and use x!). We have

$$x = t, \qquad y = t^2.$$

In coordinates this is $X = \gamma(t) = (t, t^2)$. You should verify the following formulas. Try them both by hand and by using *Mathematica*.

$$\mathbf{N} = \frac{-2t\mathbf{i}+\mathbf{j}}{\sqrt{4t^2+1}} \qquad \kappa = \frac{1}{\rho} = \frac{2}{(4t^2+1)^{3/2}}$$

$$\rho\mathbf{N} = -(4t^3+t)\mathbf{i} + (2t^2+1/2)\mathbf{j} \qquad X + \rho\mathbf{N} = (-4t^3, 3t^2+1/2)$$

The original curve is pictured along with its evolute (the curve traced out by the center of curvature). The points corresponding to $t = 0, 1$ are indicated on the curve, as are their centers of curvature and osculating circles. When $t = 0$, the point on the curve is the origin. The center of curvature is $(0, 1/2)$, the radius of curvature is $1/2$, and the curvature is 2. When $t = 1$, the point on the curve is $(1, 1)$. The center of curvature is $(-4, 7/2)$, the radius of curvature is $5\sqrt{5}/2$, and the curvature is $2/(5\sqrt{5})$.

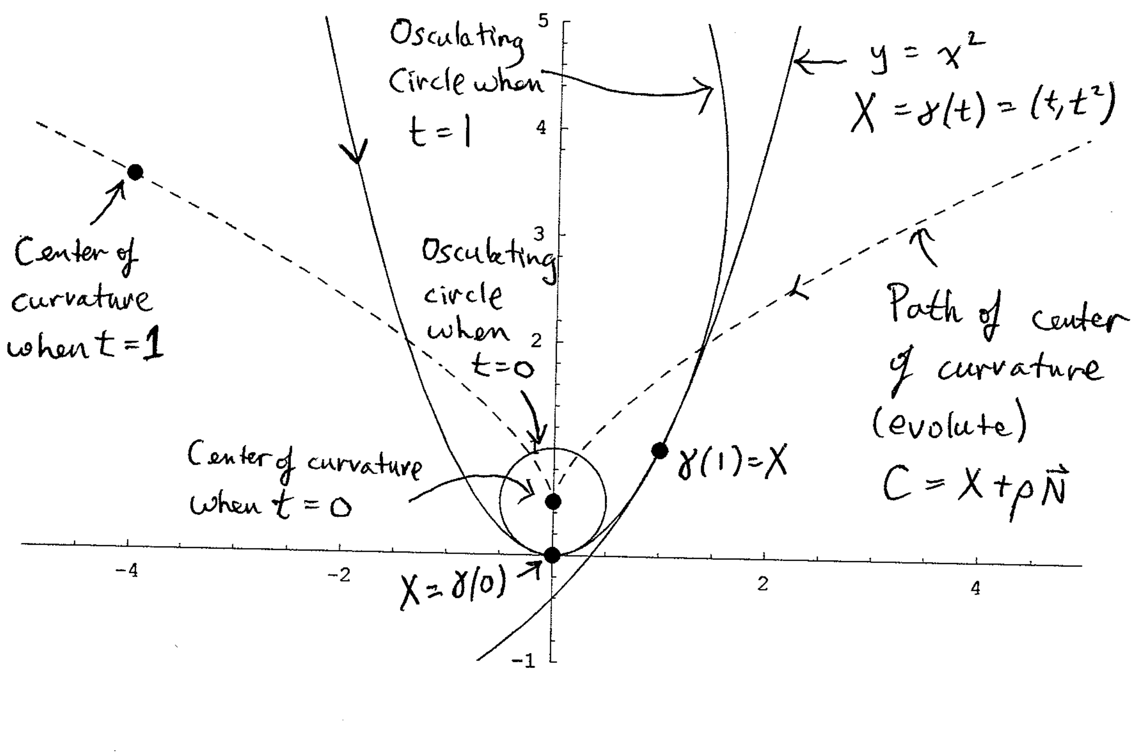
(2) It is important to realize that different orders of computation may be easier for different problems, particularly if only partial information is given. For example, suppose that the speed of the particle is $\frac{ds}{dt} = v = \sqrt{2e^{2t}+1}$ and that at $t = 0$ the acceleration is $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$. The formulas involving the vector $\mathbf{v}$ are useless since we don't know $\mathbf{v}$. Nevertheless, we can still compute the components of $\mathbf{a}$ parallel and normal to the direction of motion, that is, we can compute a_T and a_N in the expression

$$\mathbf{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N} = a_T\mathbf{T} + a_N\mathbf{N},$$

and we can compute $|\kappa|$ (there's not enough information to determine if the curve is turning clockwise or counterclockwise, but there is enough to determine how fast it is turning). We have $\frac{dv}{dt} = 4e^{2t}/\sqrt{2e^{2t}+1}$. (Note that this is *not* the magnitude of the acceleration!) At $t = 0$ we get $\frac{dv}{dt} = 4/\sqrt{3}$. From the formula $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a} = a_T^2 + a_N^2$ (why is this true?), we get

$$|a_N| = \sqrt{\|\mathbf{a}\|^2 - a_T^2},$$

Then we have $|a_N| = |\kappa|v^2 = 3|\kappa| = \sqrt{13 - 16/3} = \sqrt{69}/3$, and so $|\kappa| = \sqrt{69}/9$ at $t = 0$.



The Difference Between E^2 and E^n for $n > 2$

The theory of the geometry of curves differs slightly between E^2 and E^n, $n > 2$. The reason for this difference is essentially because the notion of "clockwise" makes sense in two dimensions, but not in higher dimensions. Our text presents the theory in higher dimensions and doesn't do the two-dimensional case separately.

The reason for doing the two-dimensional case separately is that curvature will indicate not only how fast a curve changes direction, but also whether it is turning clockwise or counterclockwise.

Given a curve, you always start by defining the unit tangent vector $\mathbf{T}$, no matter what dimension the curve is in. Then things become different.

In E^2, the unit normal vector is defined before the curvature by $\mathbf{N} = \mathbf{T}_\perp$. Then the curvature κ is defined by the equation $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$. Note that κ can be positive or negative, which distinguishes counterclockwise from clockwise.

In higher dimensions the curvature is defined first by $\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$, and so κ is non-negative. Then the unit normal vector is defined by $\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{ds}/\left\|\frac{d\mathbf{T}}{ds}\right\|$, provided κ is not zero.

Note that in all dimensions the formula $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ holds by definition, but that in two dimensions this is the definition of κ, whereas in higher dimensions it is the definition of $\mathbf{N}$.

Note also that computing $\frac{d\mathbf{T}}{ds}$ directly as a derivative is usually difficult. Fortunately it can be avoided. There are formulas for κ and $\mathbf{N}$ in terms of velocity and acceleration in E^2 and E^3 that are easier to use than the definitions. Similar formulas exist in higher dimensions.

In E^3 you need a third unit vector to make a moving frame. This vector is $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, and is called the unit bi-normal vector. The osculating circle is contained in the osculating plane, which passes through the point $X = \gamma(t)$ and is perpendicular to $\mathbf{B}$. There is also a quantity called torsion, denoted by τ, that measures how fast the curve is twisting out of the osculating plane.